HEISENBERG-LORENTZ QUANTUM GROUP

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ABSTRACT. In this article we present a new deformation of $SL(2,\mathbb{C})$ on the C*-algebra level. The method of its construction is the Rieffel deformation. We give a detailed description of so obtained a quantum group (A,Δ) in terms of affiliated elements $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}\in A^\eta$ generating A. In order to construct $\hat{\beta}$, we split A onto parts on which $\hat{\beta}$ can easily be defined and then show that gluing procedure can be performed. Using the duality for locally compact quantum groups we were able to describe all representations of the C*-algebra A and to analyze the action of Δ on generators $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$.

CONTENTS

1. INTRODUCTION

A complete classification of deformations of $SL(2,\mathbb{C})$ on the Hopf $*$-algebra level was presented in [13]. Up till now three cases contained there have been realized on a deeper, C*-algebraic level (see [2], [6], [14]). This paper is devoted to the C*-algebraic realization of another case. The method of deformation that we use is the Rieffel deformation and it is the same as in the example considered in [2]. Nevertheless the resulting quantum group $\mathbb{G}=(A,\Delta)$ is much more complex (in what follows this group will be called the Heisenberg-Lorentz quantum group). One of the difficulties lies in the fact that among the four generators $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$ of the C*-algebra A, only $\hat{\gamma}$ is normal. Also the analysis of the comultiplication Δ is not as straightforward as in the case of [2]. To perform it we use the one to one correspondence between representations of C*-algebra A and corepresentations of the dual quantum group $\hat{\mathbb{G}}$. This correspondence was also used to describe all representations of A on Hilbert spaces.

1991 *Mathematics Subject Classification.* Primary 46L89, Secondary 58B32, 22D25.
The research was supported by the KBN under grant 115/E-343/SPB/6.PR UE/DIE 50/2005 2008.

Let us briefly describe the contents of the paper. In the next section we present the Hopf $*$-algebraic version of the Heisenberg-Lorentz quantum group. We begin with a description of commutation relations satisfied by generators $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$. Formulas for the comultiplication, coinverse and counit on generators are the same as in the classical case. In Section 3 we define Hilbert space representations of the Heisenberg-Lorentz commutation relations. We show that the tensor product of two such representations can be defined. Section 4 is devoted to the construction of the C*-algebraic version (A,Δ) of the Heisenberg-Lorentz quantum group. In particular we introduce four affiliated elements $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}\in A^\eta$. The construction of the most difficult element $\hat{\beta}$ is based on the idea of splitting A into parts on which $\hat{\beta}$ has a simpler form. After defining $\hat{\beta}$ on these parts we perform a gluing procedure obtaining $\hat{\beta}\in A^\eta$ as a result. Having constructed affiliated elements $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$, we show that they generate A. Moreover, we note that for any representation $\pi\in\mathrm{Rep}(A;\mathcal{H})$ the quadruple $(\pi(\hat{\alpha}),\pi(\hat{\beta}),\pi(\hat{\gamma}),\pi(\hat{\delta}))$ is a Hilbert space representation of the Heisenberg-Lorentz commutation relations. The converse is also true: for any representation $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta})$ of the Heisenberg-Lorentz commutation relations on a Hilbert space $\mathcal{H}$ there exists a unique representation $\pi\in\mathrm{Rep}(A;\mathcal{H})$ such that

$$\pi(\hat{\alpha})=\tilde{\alpha},\;\;\pi(\hat{\beta})=\tilde{\beta},\;\;\pi(\hat{\gamma})=\tilde{\gamma},\;\;\pi(\hat{\delta})=\tilde{\delta}.$$

At the end of Section 4 we show that the action of Δ on generators has the same form as in the classical case. Appendices gather useful facts concerning the quantization map and the counit in the Rieffel deformation, the complex infinitesimal generator of the Heisenberg group and the product of strongly commuting affiliated elements.

Throughout the paper we will freely use the language of C*-algebras and the theory of locally compact quantum groups. For a locally compact space X, $\mathrm{C}_0(X)$ shall denote the algebra of continuous functions on X vanishing at infinity. If X is also a manifold, then $\mathrm{C}^\infty(X)$ denotes the algebra of smooth functions on X and $\mathrm{C}^\infty_c(X)$ denotes the algebra of smooth functions of compact supports. For the notion of multipliers, affiliated elements and algebras generated by a family of affiliated elements we refer the reader to [10], [11] and [12]. The set of elements affiliated with a C*-algebra A will be denoted by A^η. The z-transform of $T\in A^\eta$ will be denoted by z_T. For the precise definition of z_T we refer to [10]. For the theory of locally compact quantum groups we refer to [3] and [4]. For the theory of quantum groups given by a multiplicative unitary we refer to [1] and [9]. For the notion of Γ-product we refer to [5]. All Hilbert spaces appearing in the paper are assumed to be separable.

2. Hopf *-algebra level

We fix a deformation parameter $s\in\mathbb{R}$. Let $\mathcal{A}$ be a unital $*$-algebra generated by four elements $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}$ satisfying the following commutation relations:

$$\left.\begin{array}{cccc}
 & \hat{\alpha}\hat{\beta}=\hat{\beta}\hat{\alpha}\quad \hat{\alpha}\hat{\gamma}=\hat{\gamma}\hat{\alpha} & \hat{\alpha}\hat{\delta}=\hat{\delta}\hat{\alpha} & \\
 & \hat{\beta}\hat{\gamma}=\hat{\gamma}\hat{\beta} & \hat{\beta}\hat{\delta}=\hat{\delta}\hat{\beta} & \\
 & & \hat{\gamma}\hat{\delta}=\hat{\delta}\hat{\gamma} & \\
 & & \hat{\alpha}\hat{\delta}-\hat{\beta}\hat{\gamma}=1 & \\
\hat{\alpha}\hat{\alpha}^*-\hat{\alpha}^*\hat{\alpha}=-s\hat{\gamma}^*\hat{\gamma} & \hat{\alpha}\hat{\beta}^*-\hat{\beta}^*\hat{\alpha}=-s\hat{\gamma}\hat{\delta}^* & \hat{\alpha}\hat{\gamma}^*=\hat{\gamma}^*\hat{\alpha} & \hat{\alpha}\hat{\delta}^*=\hat{\delta}^*\hat{\alpha}\\
 & \hat{\beta}\hat{\beta}^*-\hat{\beta}^*\hat{\beta}=s(\hat{\alpha}^*\hat{\alpha}-\hat{\delta}\hat{\delta}^*) & \hat{\beta}\hat{\gamma}^*=\hat{\gamma}^*\hat{\beta} & \hat{\beta}\hat{\delta}^*-\hat{\delta}^*\hat{\beta}=s\hat{\gamma}^*\hat{\alpha}\\
 & & \hat{\gamma}\hat{\gamma}^*=\hat{\gamma}^*\hat{\gamma} & \hat{\gamma}\hat{\delta}^*=\hat{\delta}^*\hat{\gamma}\\
 & & & \hat{\delta}\hat{\delta}^*-\hat{\delta}^*\hat{\delta}=s\hat{\gamma}\hat{\gamma}^*.
\end{array}\right\}\quad(1)$$

The $*$-algebra $\mathcal{A}$ was introduced in [13] where it was also proven that it admits the structure of a Hopf $*$-algebra. The action of the comultiplication Δ on the generators is given by:

$$\begin{array}{c}\Delta(\hat{\alpha})=\hat{\alpha}\otimes\hat{\alpha}+\hat{\beta}\otimes\hat{\gamma}\;,\;\Delta(\hat{\beta})=\hat{\alpha}\otimes\hat{\beta}+\hat{\beta}\otimes\hat{\delta}\\ \Delta(\hat{\gamma})=\hat{\gamma}\otimes\hat{\alpha}+\hat{\delta}\otimes\hat{\gamma}\;\;,\;\Delta(\hat{\delta})=\hat{\gamma}\otimes\hat{\beta}+\hat{\delta}\otimes\hat{\delta}.\end{array}\qquad(2)$$

The coinverse κ is an involutive $*$ - antihomomorphism and its action on the generators is given by:

$$\begin{array}{l} \kappa(\hat{\alpha}) = \hat{\delta} \quad , \kappa(\hat{\beta}) = -\hat{\beta} \\ \kappa(\hat{\gamma}) = -\hat{\gamma} \ , \ \kappa(\hat{\delta}) = \hat{\alpha}. \end{array} \tag{3}$$

Finally, the action of the counit on the generators is as follows:

$$\begin{array}{l} \varepsilon(\hat{\alpha}) = 1 \ , \varepsilon(\hat{\beta}) = 0 \\ \varepsilon(\hat{\gamma}) = 0 \ , \ \varepsilon(\hat{\delta}) = 1. \end{array} \tag{4}$$

Note that the formulas defining co-operations on $\mathcal{A}$ coincide with the related formulas for the Hopf $*$-algebra of polynomial functions on $SL(2,\mathbb{C})$.

3. Hilbert space level

In this section we define a class of representations of commutation relations (1) on a Hilbert space which will be proven to correspond to representations of the C*-algebra A of the Heisenberg-Lorentz quantum group (see Theorem 4.5). Note that the pair $(\hat{\alpha}, -s\hat{\gamma}^*\hat{\gamma})$, satisfy the relation defining the lie algebra of the Heisenberg Lie group $\mathbb{H}$ (see Appendix C). The same concerns the pair $(\hat{\delta}, s\hat{\gamma}^*\hat{\gamma})$. Furthermore in the case of invertible $\hat{\gamma}$, the equation $\hat{\alpha}\hat{\delta} - \hat{\beta}\hat{\gamma} = 1$ determines $\hat{\beta}$. This gives a motivation for the following definition.

Definition 3.1. Let $\tilde{\alpha}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space $\mathcal{H}$. We say that the triple $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\delta})$ satisfies the Heisenberg-Lorentz commutation relations if:

1. $\tilde{\gamma}$ is normal and $\ker \tilde{\gamma} = \{0\}$;
2. $(\tilde{\alpha}, -s\tilde{\gamma}^*\tilde{\gamma})$ and $(\tilde{\delta}, s\tilde{\gamma}^*\tilde{\gamma})$ are infinitesimal representations of $\mathbb{H}$;
3. operators $\tilde{\alpha}, \tilde{\gamma}$ and $\tilde{\delta}$ mutually strongly commute.

For the notion of infinitesimal representation of $\mathbb{H}$ we refer to Definition C.1 and for the notion of strong commutativity we refer to Definition D.1. The above definition describes representations of commutation relations (1) in which $\hat{\gamma}$ is represented by an invertible operator $\tilde{\gamma}$. The next definition deals with the representations for which $\tilde{\gamma} = 0$. Note that in this case it is the pair $(\tilde{\beta}, s(\tilde{\alpha}^*\tilde{\alpha} - 1/\tilde{\alpha}^*\tilde{\alpha}))$ that satisfies the Heisenberg lie algebra relation.

Definition 3.2. Let $\tilde{\alpha}, \tilde{\beta}$ be closed operators acting on a Hilbert space $\mathcal{H}$. We say that the pair $(\tilde{\alpha}, \tilde{\beta})$ satisfies the Heisenberg-Lorentz commutation relations if

1. $\tilde{\alpha}$ is normal and $\ker \tilde{\alpha} = \{0\}$;
2. $(\tilde{\beta}, s(\tilde{\alpha}^*\tilde{\alpha} - 1/\tilde{\alpha}^*\tilde{\alpha}))$ is an infinitesimal representation of $\mathbb{H}$;
3. operators $\tilde{\alpha}$ and $\tilde{\beta}$ strongly commute.

Finally, using the fact that the image of $\hat{\gamma}$ has to be in the center of any representation, we may assume that any representation of Heisenberg-Lorentz commutations relations splits into a direct sum of two representations: one with an invertible $\tilde{\gamma}$ and one with $\tilde{\gamma}$ being zero. More precisely we have:

Definition 3.3. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space $\mathcal{H}$, $\tilde{\gamma}$ being normal. By $\mathcal{H}_0$, $\mathcal{H}_1$ and $\tilde{\gamma}_1$ we shall respectively denote the kernel of $\tilde{\gamma}$, its orthogonal complement and the restriction of $\tilde{\gamma}$ to $\mathcal{H}_1$. We say that the quadruple $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is a representation of the Heisenberg-Lorentz commutation relations if $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\delta}$ respect the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ i.e. there exist closed operators $\tilde{\alpha}_0, \tilde{\beta}_0, \tilde{\delta}_0$ acting on $\mathcal{H}_0$ and $\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\delta}_1$ acting on $\mathcal{H}_1$ such that

$$\tilde{\alpha} = \tilde{\alpha}_0 \oplus \tilde{\alpha}_1, \ \tilde{\beta} = \tilde{\beta}_0 \oplus \tilde{\beta}_1, \ \tilde{\delta} = \tilde{\delta}_0 \oplus \tilde{\delta}_1$$

and we have

1. the pair $(\tilde{\alpha}_0, \tilde{\beta}_0)$ satisfies the Heisenberg-Lorentz commutation relations;
2. $\tilde{\alpha}_0$ and $\tilde{\delta}_0$ are mutual inverses: $\tilde{\delta}_0 = \tilde{\alpha}_0^{-1}$;
3. the triple $(\tilde{\alpha}_1, \tilde{\gamma}_1, \tilde{\delta}_1)$ satisfies the Heisenberg-Lorentz commutation relations;
4. $\tilde{\beta}_1 = \tilde{\gamma}_1^{-1}(\tilde{\alpha}_1\tilde{\delta}_1 - 1)$.

Remark 3.4. The product of operators in point 4 above is taken in the sense of Theorem D.2. It is a well known fact that a representation of the Heisenberg group $\mathbb{H}$ can be decomposed into a direct integral of irreducible representations. In the case of irreducible representations the operator $\tilde{\gamma}$ entering Definition 3.1 and $\tilde{\alpha}$ entering Definition 3.2, are multiples of identity.

As was already mentioned, the class of representations defined above correspond to representations of the C*-algebra A of the Heisenberg-Lorentz quantum group (see Theorem 4.5). For two representations $\pi_1 \in \mathrm{Rep}(A, H_1)$ and $\pi_2 \in \mathrm{Rep}(A, H_2)$ their tensor product is given by $\pi = (\pi_1 \otimes \pi_2) \circ \Delta \in \mathrm{Rep}(A; H_1 \otimes H_2)$. In the next theorem we define the corresponding tensor product of representations of the Heisenberg-Lorentz commutation relations. This construction will be crucial in the analysis of Δ on the C*-algebra level (see Theorem (4.9)).

Theorem 3.5. *Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ be closed operators acting on a Hilbert space $\mathcal{H}$ and let $\tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}', \tilde{\delta}'$ be closed operators acting on a Hilbert space $\mathcal{H}'$. Assume that $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$, $(\tilde{\alpha}', \tilde{\beta}', \tilde{\gamma}', \tilde{\delta}')$ are representations of the Heisenberg-Lorentz commutation relations. Then the quadruple of operators $(\tilde{\alpha}'', \tilde{\beta}'', \tilde{\gamma}'', \tilde{\delta}'')$ acting on $\mathcal{H} \otimes \mathcal{H}'$, defined by:*

$$\begin{aligned} \tilde{\alpha}'' &= \tilde{\alpha} \otimes \tilde{\alpha}' \dotplus \tilde{\beta} \otimes \tilde{\gamma}' \\ \tilde{\beta}'' &= \tilde{\alpha} \otimes \tilde{\beta}' \dotplus \tilde{\beta} \otimes \tilde{\delta}' \\ \tilde{\gamma}'' &= \tilde{\gamma} \otimes \tilde{\alpha}' \dotplus \tilde{\delta} \otimes \tilde{\gamma}' \\ \tilde{\delta}'' &= \tilde{\gamma} \otimes \tilde{\beta}' \dotplus \tilde{\delta} \otimes \tilde{\delta}' \end{aligned} \tag{5}$$

is also a representation of the Heisenberg-Lorentz commutation relations.

Proof. First, let us introduce some notation. For any $\varepsilon > 0$, $z \in \mathbb{C}$ we set

$$f_\varepsilon(z) = \frac{1}{\pi\varepsilon} \exp(-\varepsilon^{-1}|z|^2) \in \mathbb{R}_+.$$

Note that $f_\varepsilon \in L^1(\mathbb{C})$ for any $\varepsilon \in \mathbb{R}_+$ and the family f_ε approximates the Dirac delta

$$\lim_{\varepsilon \to 0} \int d^2z\, f_\varepsilon(z) g(z) = g(0)$$

where d^2z is a Haar measure on $\mathbb{C}$. Let V^a be an irreducible unitary representation of $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ (for an explanation of the notation V^a we refer to the Appendix C). Smearing the family $V^a_{z,0}$ with a function f_ε we get the family of operators:

$$I^a_\varepsilon = \int d^2z\, f_\varepsilon(z) V^a_{z,0}. \tag{6}$$

The following properties of the family I^a_ε will be used in the course of the proof:

$$\left\{ \begin{array}{ll} 1. & \mathrm{s}-\lim_{\varepsilon \to 0} I^a_\varepsilon = 1, \\ 2. & \mathrm{Ran}(I_\varepsilon) \subset \mathrm{D}(a^n) \text{ for any } n \in \mathbb{N}, \\ 3. & \lim_{\varepsilon \to 0} a^n I_\varepsilon h = a^n h \text{ for any } h \in D(a^n), \end{array} \right. \tag{7}$$

$\mathrm{s}-\lim$ denotes the limit in the strong topology on $\mathrm{B}(\mathcal{H})$.

Let $c, c' \in \mathbb{C} \setminus \{0\}$. By Remark 3.4 it is enough to prove our theorem in the following three cases:

$$\left\{ \begin{array}{ll} 1. & \mathcal{H}_0 = \mathcal{H}'_0 = \{0\},\ \tilde{\gamma} = c1 \text{ and } \tilde{\gamma}' = c'1, \\ 2. & \mathcal{H}_0 = \mathcal{H}'_1 = \{0\},\ \tilde{\gamma} = c1 \text{ and } \tilde{\alpha}'_0 = c'1, \\ 3. & \mathcal{H}_1 = \mathcal{H}'_1 = \{0\},\ \tilde{\alpha}_0 = c1 \text{ and } \tilde{\alpha}'_0 = c'1. \end{array} \right. \tag{8}$$

The notation used above coincide with the notation of Definition 3.3. In what follows we shall treat case 1 leaving cases 2 and 3 to the reader. Note that the pairs $(1 \otimes c\tilde{\alpha}', -s|cc'|^2)$ and $(c'\tilde{\delta} \otimes 1, s|cc'|^2)$ are infinitesimal representations of $\mathbb{H}$. For any $z \in \mathbb{C}$ we define a unitary operator:

$$U_z = U^{1 \otimes c\tilde{\alpha}'}_{z,0} U^{c'\tilde{\delta} \otimes 1}_{z,0} \in \mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}'_1).$$

It is easy to check that the map

$$\mathbb{C} \ni z \mapsto U_z \in \mathrm{B}(\mathcal{H}_1 \otimes \mathcal{H}'_1)$$

is a strongly continuous representation of the group $(\mathbb{C},+)$. Let T be the corresponding infinitesimal generator. By definition T is a normal operator with a domain $\mathrm{D}(T)\subset\mathcal{H}_1\otimes\mathcal{H}_1'$

$$\mathrm{D}(T)=\left\{h\in\mathcal{H}_1\otimes\mathcal{H}_1':\ \begin{array}{l}\text{The map}\\ \mathbb{C}\ni z\mapsto U_z h\in\mathcal{H}_1\otimes\mathcal{H}_1'\\ \text{is once differentiable}\end{array}\right\}$$

and the action of T on $h\in\mathrm{D}(T)$ is given by

$$Th=2\frac{\partial}{\partial z}U_z h\Big|_{z=0}. \tag{9}$$

With this definition of T we have $U_z=e^{i\Im(zT)}$, which explains the factor 2 on the right hand side of (9). By formula (102) we see that $\tilde{\gamma}''\subset T$. To prove the equality $\tilde{\gamma}''=T$ it is enough to show that the linear subspace $\mathrm{D}(1\otimes\tilde{\alpha}')\cap\mathrm{D}(\tilde{\delta}\otimes 1)\subset\mathcal{H}_1\otimes\mathcal{H}_1'$ (which in the case of point 1 of (8) is a core of $\tilde{\gamma}''$) is a core of T. For the proof of this we use the family of operators $I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'}\in\mathrm{B}(\mathcal{H}_1\otimes\mathcal{H}_1')$. It has the following properties:

$$\left\{\begin{array}{ll}1. & \mathrm{s}-\lim_{\varepsilon\to 0}(I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'})=1,\\ 2. & \mathrm{Ran}(I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'})\subset\mathrm{D}(\tilde{\delta}\otimes 1)\cap\mathrm{D}(1\otimes\tilde{\alpha}'),\\ 3. & \lim_{\varepsilon\to 0}T(I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'})h=Th \text{ for any } h\in D(T).\end{array}\right. \tag{10}$$

Properties 1 and 2 above are direct consequences of (7). Property 3 demands a separate proof which is based on formulas (6) and (9). The fact that $\mathrm{D}(1\otimes\tilde{\alpha}')\cap\mathrm{D}(\tilde{\delta}\otimes 1)$ is a core of T is an immediate consequence of 1, 2 and 3, hence we have $T=\tilde{\gamma}''$. Let us show that $\ker\tilde{\gamma}''=\{0\}$. Assume that $\ker\tilde{\gamma}''\neq\{0\}$. Using the following identity:

$$\left(U^{\tilde{\delta}}_{\frac{z}{c'},0}\otimes U^{\tilde{\alpha}'}_{-\frac{z}{c},0}\right)\tilde{\gamma}''\left(U^{\tilde{\delta}}_{-\frac{z}{c'},0}\otimes U^{\tilde{\alpha}'}_{\frac{z}{c},0}\right)=\tilde{\gamma}''+\bar{z} \tag{11}$$

one immediately shows that $\tilde{\gamma}''$ has an eigenvector for any complex number. This fact and the normality of $\tilde{\gamma}''$ (eigenvectors of different eigenvalues are perpendicular) contradicts separability of $\mathcal{H}_1\otimes\mathcal{H}_1'$. This shows that $\ker\tilde{\gamma}''=\{0\}$.

Let us pass on to the analysis of the operator $\tilde{\alpha}''=\tilde{\alpha}\otimes\tilde{\alpha}'\dotplus\tilde{\beta}\otimes\tilde{\gamma}'$. Our objective is to show that $\tilde{\alpha}''$ is an infinitesimal complex generator of a representation of group $\mathbb{H}$ (cf. Definition 3.1). We define an auxiliary operator

$$T'=\tilde{\gamma}''(c^{-1}\tilde{\alpha}\otimes 1)-c^{-1}c'. \tag{12}$$

Note that $\tilde{\gamma}''$ and $\tilde{\alpha}\otimes 1$ strongly commute and by Theorem D.2, T' is well defined. It is easy to see that $(T',-s\tilde{\gamma}''^*\tilde{\gamma}'')$ is an infinitesimal representation of $\mathbb{H}$. Hence, to prove that $(\tilde{\alpha}'',-s\tilde{\gamma}''^*\tilde{\gamma}'')$ is also an infinitesimal representation of $\mathbb{H}$ it is enough to show that $\tilde{\alpha}''=T'$. For this purpose we consider the family of operators $I_\varepsilon\in\mathrm{B}(\mathcal{H}_1\otimes\mathcal{H}_1')$:

$$I_\varepsilon=I_\varepsilon^{\tilde{\alpha}}I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'}.$$

It has the following properties:

$$\left\{\begin{array}{ll}1. & \mathrm{s}-\lim_{\varepsilon\to 0}I_\varepsilon=1,\\ 2. & \mathrm{Ran}(I_\varepsilon)\subset\mathrm{D}(\tilde{\alpha}'')\cap\mathrm{D}(T'),\\ 3. & T'|_{\mathrm{Ran}(I_\varepsilon)}=\tilde{\alpha}''|_{\mathrm{Ran}(I_\varepsilon)},\\ 4. & \lim_{\varepsilon\to 0}T'I_\varepsilon h=T'h \text{ for any } h\in\mathrm{D}(T'),\\ 5. & \lim_{\varepsilon\to 0}\tilde{\alpha}''I_\varepsilon h=\tilde{\alpha}''h \text{ for any } h\in\mathrm{D}(\tilde{\alpha}'').\end{array}\right. \tag{13}$$

Properties 3, 4 and 5 show that T' and $\tilde{\alpha}''$ coincide on a join core, hence $T'=\tilde{\alpha}''$.

Similarly we prove that the operator $\tilde{\delta}''=\tilde{\gamma}\otimes\tilde{\beta}'\dotplus\tilde{\delta}\otimes\tilde{\delta}'$ gives rise to the infinitesimal representation $(\tilde{\delta}'',-s\tilde{\gamma}''^*\tilde{\gamma}'')$ of $\mathbb{H}$. To complete the proof we have to show that the operator $\tilde{\beta}''=\tilde{\alpha}\otimes\tilde{\beta}'\dotplus\tilde{\beta}\otimes\tilde{\delta}'$ is equal to $\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)$. For this purpose we define the family of operators $J_\varepsilon\in\mathrm{B}(\mathcal{H}_1\otimes\mathcal{H}_1')$:

$$J_\varepsilon=I_\varepsilon^{\tilde{\alpha}}I_\varepsilon^{\tilde{\delta}}\otimes I_\varepsilon^{\tilde{\alpha}'}I_\varepsilon^{\tilde{\delta}'}.$$

It has the following properties:

$$\left\{\begin{array}{ll} 1. & s-\lim_{\varepsilon\to 0} J_\varepsilon = 1, \\ 2. & \mathrm{Ran}(J_\varepsilon) \subset \mathrm{D}(\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)) \cap \mathrm{D}(\tilde{\beta}''), \\ 3. & \tilde{\beta}''|_{\mathrm{Ran}(J_\varepsilon)} = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)|_{\mathrm{Ran}(J_\varepsilon)}, \\ 4. & \lim_{\varepsilon\to 0} \tilde{\beta}'' J_\varepsilon h = \tilde{\beta}'' h \text{ for any } h \in \mathrm{D}(\tilde{\beta}''), \\ 5. & \lim_{\varepsilon\to 0} \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1) J_\varepsilon h = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1) h \\ & \text{for any } h \in \mathrm{D}(\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)). \end{array}\right. \tag{14}$$

Properties 3, 4 and 5 show that $\tilde{\beta}''$ and $\tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)$ coincide on a join core, therefore $\tilde{\beta}'' = \tilde{\gamma}''^{-1}(\tilde{\alpha}''\tilde{\delta}''-1)$. □

4. C*-algebra level

In this section we shall describe the Heisenberg-Lorentz quantum group on the C*-algebra level. It is obtained by applying the Rieffel deformation to $SL(2,\mathbb{C})$ (in what follows we shall denote this group by G). Let us fix an abelian subgroup $\Gamma \subset G$ which is used in the deformation procedure. We choose:

$$\Gamma = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Note that Γ is isomorphic with the additive group of complex numbers. In particular Γ and its Pontryiagin dual $\hat{\Gamma}$ are isomorphic. The isomorphism that we shall use is given by the following non-degenerate pairing on $\mathbb{C}^2$:

$$\langle z, z' \rangle = \exp(i\Im(zz')).$$

Let Ψ be the skew bicharacter on $\hat{\Gamma} \simeq \mathbb{C}$ given by:

$$\Psi(z,z') = \exp\left(i\frac{s}{4}\Im(z\bar{z}')\right). \tag{15}$$

Using the results of Rieffel (see [8]) we know that the abelian subgroup $\Gamma \subset G$ and the bicharacter Ψ on $\hat{\Gamma}$ give rise to a quantum group $\mathbb{G} = (A, \Delta)$. In this paper we shall use a formulation of the Rieffel deformation based on the theory of crossed products that we described in [2]. In this framework the deformation procedure goes as follows. Let $\rho : \Gamma^2 \to \mathrm{Aut}(\mathrm{C}_0(G))$ be the action of Γ^2 given by the left and right shifts of functions on G along Γ:

$$\rho_{\gamma_1,\gamma_2}(f)(g) = f(\gamma_1^{-1} g \gamma_2)$$

where $\gamma_1, \gamma_2 \in \Gamma$, $f \in \mathrm{C}_0(G)$ and $g \in G$. We construct the crossed product C*-algebra $B = \mathrm{C}_0(G) \rtimes_\rho \Gamma^2$. Let $(B, \lambda, \hat{\rho})$ be the standard Γ^2-product structure on B, i.e. $\hat{\rho}$ is the dual action of $\hat{\Gamma}^2$ on B and λ is the representation of Γ^2 on B, which implements ρ on $\mathrm{C}_0(G) \subset \mathrm{M}(B)$:

$$\rho_{\gamma_1,\gamma_2}(f) = \lambda_{\gamma_1,\gamma_2} f \lambda^*_{\gamma_1,\gamma_2}.$$

This Γ^2-product will be denoted by $\mathbb{B}$. Let $\hat{\Psi} : \hat{\Gamma} \to \Gamma$ be the homomorphism given by $\hat{\Psi}(\hat{\gamma}) = \frac{s}{4}\overline{\hat{\gamma}}$ for any $\hat{\gamma} \in \hat{\Gamma}$. Using $\hat{\Psi}$ we twist $\hat{\rho}$ getting the dual action $\hat{\rho}^\Psi : \hat{\Gamma}^2 \to \mathrm{Aut}(B)$ (in [2] it was denoted by $\hat{\rho}^{\hat{\Psi}\otimes\Psi}$):

$$\hat{\rho}^\Psi_{\hat{\gamma},\hat{\gamma}'}(b) = \lambda_{-\hat{\Psi}(\hat{\gamma}),\hat{\Psi}(\hat{\gamma}')} \hat{\rho}_{\hat{\gamma},\hat{\gamma}'}(b) \lambda^*_{-\hat{\Psi}(\hat{\gamma}),\hat{\Psi}(\hat{\gamma}')}. \tag{16}$$

As was shown in [2], the triple $\mathbb{B}^\Psi = (B, \lambda, \hat{\rho}^\Psi)$ is also a Γ^2-product. The C*-algebra A of the Heisenberg-Lorentz quantum group $\mathbb{G}$ is defined as the Landstad algebra of $\mathbb{B}^\Psi$:

$$A = \left\{ b \in \mathrm{M}(B) : \begin{array}{ll} 1. & \hat{\rho}^\Psi_{\hat{\gamma},\hat{\gamma}'}(b) = b \text{ for all } \hat{\gamma}, \hat{\gamma}' \in \hat{\Gamma}; \\ 2. & \text{the map } \Gamma^2 \ni (\gamma,\gamma') \mapsto \lambda_{\gamma,\gamma'} b \lambda^*_{\gamma,\gamma'} \in \mathrm{M}(B) \\ & \text{is norm continuous;} \\ 3. & xbx' \in B \text{ for all } x, x' \in \mathrm{C}^*(\Gamma) \subset \mathrm{M}(B). \end{array} \right\} \tag{17}$$

Conditions 1, 2 and 3 will be refereed to as the Landstad conditions. The C*-algebra A carries the structure of a quantum group. All structure maps can be described in terms of the Γ^2-product $\mathbb{B}$ but in this paper we shall rather use the fact that they are related to a manageable multiplicative unitary W (for the proof of this we refer to [2]). The construction of W goes as follows. Let dg be

the right Haar measure on G and $L^2(G)$ be the Hilbert space of square integrable functions with respect to dg. Let $R_g, L_g \in \mathrm{B}(L^2(G))$ be the right and left regular representation of G. Restricting them to $\Gamma \subset G$ we get two representations of Γ on $L^2(G)$. The induced representations of $\mathrm{C}^*(\Gamma)$ will be respectively denoted by $\pi^R, \pi^L \in \mathrm{Rep}(\mathrm{C}^*(\Gamma); L^2(G))$. Obviously $\Psi \in \mathrm{M}(\mathrm{C}^*(\Gamma) \otimes \mathrm{C}^*(\Gamma))$ is unitary, hence operators $X, Y \in \mathrm{B}(L^2(G) \otimes L^2(G))$ given by

$$\begin{aligned} X &= (\pi^R \otimes \pi^R)(\Psi), \\ Y &= (\pi^R \otimes \pi^L)(\Psi) \end{aligned} \tag{18}$$

are unitary too. Finally, the multiplicative unitary $W \in \mathrm{B}(L^2(G) \otimes L^2(G))$ related to A is of the following form:

$$W = YVX \tag{19}$$

where V is the Kac-Takesaki operator of the group G. The C^*-algebra A is isomorphic with the C^*-algebra of slices of the first leg of W (which is contained in $\mathrm{B}(L^2(G))$):

$$A \simeq \left\{ (\omega \otimes \mathrm{id})W : \omega \in \mathrm{B}(L^2(G))_* \right\}^{\|\cdot\|-\text{closure}}. \tag{20}$$

Note that A treated as the algebra of slices of W is naturally represented on $L^2(G)$.

4.1. **Affiliated element** $\hat{\gamma}$**.** Let $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ be the identity function: $\mathrm{id}(z) = z$ for any $z \in \mathbb{C}$. This function generates $\mathrm{C}_0(\mathbb{C})$ in the sense of Woronowicz (see Definition 3.1 of [11]). Let $\pi \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{C}); \mathrm{C}_0(G))$ be the morphism that sends $\mathrm{id} \in \mathrm{C}_0(\mathbb{C})^\eta$ to the coordinate function $\gamma \in \mathrm{C}_0(G)^\eta$. From the invariance of γ under the action $\rho : \Gamma^2 \to \mathrm{Aut}(\mathrm{C}_0(G))$ it follows that π satisfies the assumptions of Theorem 3.18 of [2] for the trivial action of Γ^2 on $\mathrm{C}_0(\mathbb{C})$, therefore it gives rise to the twisted morphism $\pi^\Psi \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{C}); A)$. We define $\hat{\gamma} \in A^\eta$ as the image of $\mathrm{id} \in \mathrm{C}_0(\mathbb{C})^\eta$ under π^Ψ: $\hat{\gamma} = \pi^\Psi(\mathrm{id})$. One can check that the z-transform $z_{\hat{\gamma}}$ belongs to the center of $M(A)$ and $\hat{\gamma}$ treated as an operator acting on $L^2(G)$ (c.f. (20)) coincides with the operator of multiplication by the coordinate γ.

4.2. **Affiliated elements** $\hat{\alpha}$ **and** $\hat{\delta}$**.** Let us fix some notation. Let T be a normal element affiliated with a C^*-algebra C. By $\lambda(z; T)$ we shall denote the unitary given by:

$$\lambda(z; T) = \exp(i\Im(zT)) \in \mathrm{M}(C). \tag{21}$$

Note that the map

$$\mathbb{C} \ni z \mapsto \lambda(z; T) \in \mathrm{M}(C)$$

is a representation of the additive group $\mathbb{C}$. We shall denote this representation by $\lambda(\,\cdot\,; T)$.

Let $\mathbb{B} = (B, \lambda, \hat{\rho})$ be the Γ^2-product introduced above and let $T_l, T_r \in B^\eta$ be infinitesimal generators of representation $\lambda : \mathbb{C}^2 \to \mathrm{M}(B)$:

$$\lambda_{z_1,z_2} = \lambda(z_1, T_l)\lambda(z_2, T_r). \tag{22}$$

Note that the coordinate function $\gamma \in \mathrm{C}_0(G)^\eta \subset B^\eta$ commutes with λ_{z_1,z_2}:

$$\lambda_{z_1,z_2}\gamma\lambda^*_{z_1,z_2} = \gamma.$$

This implies that T_l, T_r and γ strongly commute in the sense of Definition D.1. Using Theorem D.2 we construct a pair of normal elements: $\gamma T_l^*, \gamma T_r^* \in B^\eta$. Let us also consider normal elements $\alpha, \delta, \gamma^*\gamma \in \mathrm{C}_0(G)^\eta \subset B^\eta$. For any $(z, t) \in \mathbb{H}$ we define two unitaries in $\mathrm{M}(B)$:

$$\begin{aligned} U^{\hat{\alpha}}_{z,t} &= \lambda\left(z; \alpha\right) \lambda\left(z; -\tfrac{s}{4}\gamma T_l^*\right) \lambda\left(t; -\tfrac{s}{4}\gamma^*\gamma\right), \\ U^{\hat{\delta}}_{z,t} &= \lambda\left(z; \delta\right) \lambda\left(z; -\tfrac{s}{4}\gamma T_r^*\right) \lambda\left(t; \tfrac{s}{4}\gamma^*\gamma\right). \end{aligned} \tag{23}$$

Let us introduce on G two complex vector fields ∂_l and ∂_r:

$$\begin{aligned} \partial_l f(g) &= 2\tfrac{\partial}{\partial z} L_z(f)(g)|_{z=0}, \\ \partial_r f(g) &= 2\tfrac{\partial}{\partial z} R_z(f)(g)|_{z=0} \end{aligned} \tag{24}$$

for any $f \in \mathrm{C}^\infty(G)$ and any $g \in G$ (L_z and R_z in (24) denote operators of the left and right shift by an element $z \in \Gamma$). Using ∂_l and ∂_R we define differential operators $\mathrm{Op}(\alpha)$ and $\mathrm{Op}(\delta)$ acting on $\mathrm{C}^\infty_c(G)$:

$$\begin{array}{l}\mathrm{Op}(\alpha)=\alpha-\frac{s}{4}\gamma\partial_l^*,\\ \mathrm{Op}(\delta)=\delta-\frac{s}{4}\gamma\partial_r^*\end{array}. \tag{25}$$

The quantization map $\mathcal{Q}$ used in the next theorem is described in Appendix A.

Theorem 4.1. *Let $U^{\hat\alpha}_{z,t},U^{\hat\delta}_{z,t}\in\mathrm{M}(B)$ be the unitary elements given by (23). Let $\mathrm{Op}(\alpha)$ and $\mathrm{Op}(\delta)$ be the differential operators given by (25) and A the Landstad algebra of $\mathbb{B}^\Psi$ (see (17)). Then:*

1. *$U^{\hat\alpha}_{z,t},U^{\hat\delta}_{z,t}$ are elements of $\mathrm{M}(A)\subset\mathrm{M}(B)$;*
2. *the maps*
$$\mathbb{H}\ni(z,t)\mapsto U^{\hat\alpha}_{z,t}\in\mathrm{M}(A)$$
$$\mathbb{H}\ni(z,t)\mapsto U^{\hat\delta}_{z,t}\in\mathrm{M}(A)$$
are strictly continuous, commuting representations of the Heisenberg group;
3. *Let $\hat\alpha,\hat\delta\in A^\eta$ be complex generators of $U^{\hat\alpha}$ and $U^{\hat\delta}$. The set $\{\mathcal{Q}(f):f\in\mathrm{C}_c^\infty(G)\}\subset A$ is a common core of $\hat\alpha,\hat\delta\in A^\eta$ and we have*
$$\begin{array}{l}\hat\alpha\mathcal{Q}(f)=\mathcal{Q}(\mathrm{Op}(\alpha)f)\\ \hat\delta\mathcal{Q}(f)=\mathcal{Q}(\mathrm{Op}(\delta)f)\end{array} \tag{26}$$
for any $f\in\mathrm{C}_c^\infty(G)$.

Proof. Let us begin by proving that $U^{\hat\alpha}_{z,t},U^{\hat\delta}_{z,t}\in\mathrm{M}(A)$. Let $\hat\rho^\Psi$ be the twisted dual action given by (16). It is easy to check that

$$\left\{\begin{array}{ll}1. & \hat\rho^\Psi_{z_1,z_2}(\lambda(z;\alpha))=\lambda\left(z,\frac{s}{4}\bar z_1\gamma\right)\lambda(z;\alpha),\\ 2. & \hat\rho^\Psi_{z_1,z_2}\left(\lambda\left(z;-\frac{s}{4}\gamma T_l^*\right)\right)=\lambda(z;-\frac{s}{4}\bar z_1\gamma)\lambda\left(z;-\frac{s}{4}\gamma T_l^*\right),\\ 3. & \hat\rho^\Psi_{z_1,z_2}(\lambda(z;\delta))=\lambda\left(z;\frac{s}{4}\bar z_2\gamma\right)\lambda(z;\delta),\\ 4. & \hat\rho^\Psi_{z_1,z_2}\left(\lambda\left(z;-\frac{s}{4}\gamma T_r^*\right)\right)=\lambda(z;-\frac{s}{4}\bar z_2\gamma)\lambda\left(z;-\frac{s}{4}\gamma T_r^*\right).\end{array}\right.$$

Using these equalities and (23) we see that $U^{\hat\alpha}_{z,t}$ and $U^{\hat\delta}_{z,t}$ satisfy the first Landstad condition (see (17)). Let us pass on to the second Landstad condition. One can check that:

$$\lambda_{z_1,z_2}U^{\hat\alpha}_{z,t}\lambda^*_{z_1,z_2}=\lambda(z,-z_1\gamma)U^{\hat\alpha}_{z,t},$$
$$\lambda_{z_1,z_2}U^{\hat\delta}_{z,t}\lambda^*_{z_1,z_2}=\lambda(z,z_2\gamma)U^{\hat\alpha}_{z,t}.$$

In Section 4.1 we constructed the affiliated element $\hat\gamma\in A^\eta$. Its image in B^η coincides with the coordinate function $\gamma\in\mathrm{C}_0(G)^\eta\subset B^\eta$. Therefore, the following two maps are norm continuous:

$$\mathbb{C}^2\ni(z_1,z_2)\mapsto\lambda_{z_1,z_2}U^{\hat\alpha}_{z,t}\lambda^*_{z_1,z_2}a\in\mathrm{M}(B)$$
$$\mathbb{C}^2\ni(z_1,z_2)\mapsto\lambda_{z_1,z_2}U^{\hat\delta}_{z,t}\lambda^*_{z_1,z_2}a\in\mathrm{M}(B).$$

for any $a\in A$. This norm-continuity together with $\hat\rho^\Psi$-invariance of $U^{\hat\alpha}_{z,t},U^{\hat\delta}_{z,t}\in\mathrm{M}(B)$ implies that $U^{\hat\alpha}_{z,t},U^{\hat\delta}_{z,t}$ are indeed elements of $\mathrm{M}(A)$. The proof that $U^{\hat\alpha}$ and $U^{\hat\delta}$ are commuting representations of $\mathbb{H}$ is left to the reader. Let us now prove the strict continuity of these representations. For this purpose we shall treat $U^{\hat\alpha}_{z,t}$ and $U^{\hat\delta}_{z,t}$ as operators acting on $L^2(G)$. It can be checked that the action of $U^{\hat\alpha}$ and $U^{\hat\delta}$ on $L^2(G)$ expressed in coordinates α,δ,γ is given by:

$$\begin{array}{l}U^{\hat\alpha}_{z,t}f(\alpha,\gamma,\delta)=\exp\left(-i\frac{st}{4}\bar\gamma\gamma\right)\exp(i\Im(z\alpha))f(\alpha-\frac{s}{4}\bar z\bar\gamma\gamma,\gamma,\delta)\\ U^{\hat\delta}_{z,t}f(\alpha,\gamma,\delta)=\exp\left(i\frac{st}{4}\bar\gamma\gamma\right)\exp(i\Im(z\delta))f(\alpha,\gamma,\delta+\frac{s}{4}\bar z\bar\gamma\gamma)\end{array} \tag{27}$$

for any $f\in L^2(G)$. Using Theorem 4.14 of [2] we obtain:

$$U^{\hat\alpha}_{z,t}\mathcal{Q}(f)=\mathcal{Q}(U^{\hat\alpha}_{z,t}f) \tag{28}$$

for any $f\in\mathrm{C}_c^\infty(G)$. This together with Theorem A.1 leads to the following estimation:

$$\|U^{\hat\alpha}_{z,t}\mathcal{Q}(f)-\mathcal{Q}(f)\|\leq c\max_{k,k',l,l'\leq5}\sup_{g\in G}\left|\partial_l^{k*}\partial_l^{k'}\partial_r^{*m}\partial_r^{m'}(U^{\hat\alpha}_{z,t}f-f)\right|.$$

The right hand side of the above inequality is easily checked to be convergent to zero when $(z,t) \to (0,0)$, which shows that:

$$\lim_{(z,t)\to(0,0)} \|U^{\hat{\alpha}}_{z,t}\mathcal{Q}(f) - \mathcal{Q}(f)\| = 0.$$

The density of the set $\{\mathcal{Q}(f) : f \in \mathrm{C}^\infty_c(G)\}$ in A ensures that $U^{\hat{\alpha}}$ is continuous in the sense of the strict topology of $\mathrm{M}(A)$. Similarly, we prove the strict continuity of the representation $U^{\hat{\delta}}$.

Let us pass on to the proof of the third point of our theorem. Using equation (27) one can check that

$$2\frac{\partial}{\partial z}U^{\hat{\alpha}}_{z,0}f\Big|_{z=0} = \mathrm{Op}(\alpha)f$$

for any $f \in \mathrm{C}^\infty_c(G)$. This together with (28) and (103) proves the first equation of (26). The second formula of (26) is proof in a similar way. To prove that $\mathcal{Q}(\mathrm{C}^\infty_c(G))$ is a core of either $\hat{\alpha}$ or $\hat{\delta}$ it is enough to check that the sets

$$\begin{gathered}\{(1+\hat{\alpha}^*\hat{\alpha})\mathcal{Q}(f) : f \in \mathrm{C}^\infty_c(G)\},\\ \{(1+\hat{\delta}^*\hat{\delta})\mathcal{Q}(f) : f \in \mathrm{C}^\infty_c(G)\}\end{gathered} \tag{29}$$

are dense in A (see Lemma D.3). In what follows we shall sketch the prove of the density for the first of these sets. Let us first note that the set

$$(1+\hat{\alpha}^*\hat{\alpha})^{-1}\mathcal{Q}(\mathrm{C}^\infty_c(G)) = \{(1+\hat{\alpha}^*\hat{\alpha})^{-1}\mathcal{Q}(f) : f \in \mathrm{C}^\infty_c(G)\} \subset A$$

is dense in A. This follows from the density of $\mathcal{Q}(\mathrm{C}^\infty_c(G))$ in A and the fact that $\hat{\alpha} \in A^\eta$. Let f be an arbitrary element of $\mathrm{C}^\infty_c(G)$ and $g \in L^2(G)$ the function given by:

$$g = (1+\hat{\alpha}^*\hat{\alpha})^{-1}f.$$

Using (101) we see that

$$g = \int_{\mathbb{R}_+} dt\, \exp(-t) \int_{\mathbb{C}} d^2z\, h_t\left(z, -s\frac{1}{2}\hat{\gamma}^*\hat{\gamma}\right) U^{\hat{\alpha}}_{z,0}f. \tag{30}$$

One can check that g is quantizable in the sense of Theorem A.1, $\mathcal{Q}(g) \in \mathrm{D}(\hat{\alpha}^*\hat{\alpha})$ and $\mathcal{Q}(f) = (1+\hat{\alpha}^*\hat{\alpha})\mathcal{Q}(g)$. Using formula (30) we can prove the existence of a sequence $f_n \in \mathrm{C}^\infty_c(G)$ such that

$$\begin{gathered}\lim_{n\to\infty} \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}f_n = \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}g\\ \lim_{n\to\infty} \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}(1+\mathrm{Op}(\alpha)^*\,\mathrm{Op}(\alpha))f_n = \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}(1+\mathrm{Op}(\alpha)^*\,\mathrm{Op}(\alpha))g\end{gathered} \tag{31}$$

for any $k, k', m, m' \leq 5$. By the equations (26), (31), Theorem (A.1) and the closedness of $\hat{\alpha}$ we get

$$\mathcal{Q}(f) = \lim_{n\to\infty}(1+\hat{\alpha}^*\hat{\alpha})\mathcal{Q}(f_n).$$

Using the fact that f is an arbitrary smooth function of compact support and $\mathcal{Q}(\mathrm{C}^\infty_c(G))$ is a dense subset of A we get:

$$\overline{(1+\hat{\alpha}^*\hat{\alpha})\mathcal{Q}(\mathrm{C}^\infty_c(G))}^{\|\cdot\|} = A.$$

This ends the proof of the density of the set (29) for $\hat{\alpha}$. □

4.3. **Quantum Borel subgroup $\mathbb{G}_0$.** Let $G_0 \subset G$ be a Borel subgroup of G:

$$G_0 = \left\{\begin{pmatrix}\alpha_0 & \beta_0\\ 0 & \alpha_0^{-1}\end{pmatrix} : \alpha_0 \in \mathbb{C}_*, \beta_0 \in \mathbb{C}\right\}$$

and $\pi_0 \in \mathrm{Mor}(\mathrm{C}_0(G); \mathrm{C}_0(G_0))$ be the restriction morphism to $G_0 \subset G$:

$$\pi_0(f)(g_0) = f(g_0)$$

for any $f \in \mathrm{C}_0(G)$ and $g_0 \in G_0$. Note that $\Gamma \subset G_0$. Applying the Rieffel deformation to $(\mathrm{C}_0(G_0), \Delta)$, based on the subgroup Γ we construct a quantum group $\mathbb{G}_0 = (A_0, \Delta)$. Let $\mathbb{B}_0$ be the respective Γ^2-product. In this section T_l and T_r denote the infinitesimal generators of the representation $\lambda : \mathbb{C}^2 \to \mathrm{M}(B_0)$ (c.f. (22)) and ∂_l, ∂_r denote the vector fields on G_0 defined analogously to (24). By Theorem 3.18 of [2] the restriction morphism $\pi_0 \in \mathrm{Mor}(\mathrm{C}_0(G); \mathrm{C}_0(G_0))$

induces the twisted morphism of C*-algebras $\pi_0^\Psi : A \to A_0$ and the surjectivity of π_0 implies the surjectivity of π_0^Ψ. Let $A_{\hat\gamma} \subset A$ be the two-sided ideal generated by $z_{\hat\gamma}$. Invoking the centrality of $z_{\hat\gamma}$ in $\mathrm{M}(A)$ we have $A_{\hat\gamma} = \overline{z_{\hat\gamma} A}^{\|\cdot\|}$. It is easy to see that $\pi_0^\Psi(z_{\hat\gamma}) = 0$, which implies that $A_{\hat\gamma} \subset \ker \pi_0^\Psi$. It can also be proven that $\ker \pi_0^\Psi \subset A_{\hat\gamma}$, hence we have the exact sequence of C*-algebras:

$$0 \to A_{\hat\gamma} \to A \xrightarrow{\pi_0^\Psi} A_0 \to 0. \tag{32}$$

This sequence will be very useful in the construction of $\hat\beta \in A^\eta$.

In what follows we shall construct an affiliated element $\hat\beta_0 \in A_0^\eta$, which is farther used in the construction of $\hat\beta \in A^\eta$. Let us first mention that following the construction of $\hat\gamma \in A^\eta$ of Section 4.1, we may introduce an affiliated element $\hat\alpha_0 \in A_0^\eta$. It is normal and invertible and its action on $L^2(G_0)$ is given by the multipliciation operator by the coordinate α_0. Remebering that $\mathrm{C}_0(G_0)^\eta \subset B_0^\eta$ we shall consider α_0 and β_0 as affiliated with B_0. Note that α_0, T_l and T_r strongly commute. Using Theorem D.2 we construct $\alpha_0 T_r^*$, $\alpha_0^{-1} T_l^* \in B_0$. For any $(z,t) \in \mathbb{H}$ we define the unitary element:

$$U_{z,t}^{\hat\beta_0} = \lambda\,(z;\beta_0)\,\lambda\left(z; -\frac{s}{4}\alpha_0^{-1}T_l^*\right)\lambda\left(z; -\frac{s}{4}\alpha_0 T_r^*\right)\lambda\left(t; -\frac{s}{4}(|\alpha_0|^{-2} - |\alpha_0|^2)\right) \in \mathrm{M}(B_0). \tag{33}$$

Let us also define the differential operator:

$$\mathrm{Op}(\beta_0) = \beta_0 - \frac{s}{4}\alpha_0^{-1}\partial_l^* - \frac{s}{4}\alpha_0\partial_r^*. \tag{34}$$

The proof of the next theorem is analogous to the proof of Theorem 4.1. The quantization map related to the quantum group $\mathbb{G}_0$ is denoted by $\mathcal{Q}_0$.

Theorem 4.2. *Let $U_{z,t}^{\hat\beta_0} \in \mathrm{M}(B_0)$ be the unitary element given by formula (33). Then*

1. *$U_{z,t}^{\hat\beta_0}$ is an element of $\mathrm{M}(A_0) \subset \mathrm{M}(B_0)$ for any $(z,t) \in \mathbb{H}$;*
2. *the map*
$$\mathbb{H} \ni (z,t) \mapsto U_{z,t}^{\hat\beta_0} \in \mathrm{M}(A_0)$$
is a strongly continuous representation of the Heisenberg group;
3. *the set $\{\mathcal{Q}_0(f) : f \in \mathrm{C}_c^\infty(G_0)\} \subset A_0$ is a core of the generator $\hat\beta_0 \in A_0^\eta$ of representation $U^{\hat\beta_0}$ and we have*
$$\hat\beta_0 \mathcal{Q}_0(f) = \mathcal{Q}_0(\mathrm{Op}(\beta_0)f) \tag{35}$$
for any $f \in \mathrm{C}_c^\infty(G_0)$. Moreover the set
$$\{\mathcal{Q}_0((1 + \mathrm{Op}(\beta_0)^*\,\mathrm{Op}(\beta_0))f) : f \in \mathrm{C}_c^\infty(G_0)\} \tag{36}$$
is dense in A_0.

4.4. **Affiliated element $\hat\beta$.** Let $\pi_{\hat\gamma} \in \mathrm{Mor}(A, A_{\hat\gamma})$ be the morphism defined by the formula $\pi_{\hat\gamma}(a)a_{\hat\gamma} = a a_{\hat\gamma}$ for any $a \in A$ and $a_{\hat\gamma} \in A_{\hat\gamma}$. This morphism is injective, which enables us to treat $\hat\alpha, \hat\gamma, \hat\delta \in A^\eta$ as elements affiliated with $A_{\hat\gamma}$. The injectivity of $\pi_{\hat\gamma}$ follows from the implication $(a z_{\hat\gamma} = 0) \Rightarrow (a = 0)$, which is true for any $a \in A$. Note that $\hat\gamma$ treated as an element of $A_{\hat\gamma}$ is invertible, i.e. there exists a unique element $\hat\gamma^{-1} \in A_{\hat\gamma}^\eta$ strongly commuting with $\hat\gamma$ and such that $\hat\gamma\hat\gamma^{-1} = \hat\gamma^{-1}\hat\gamma = 1$. Moreover, elements $\hat\alpha, \hat\delta, \hat\gamma^{-1} \in A_{\hat\gamma}^\eta$ mutually strongly commute and using Theorem D.2 we define

$$\hat\beta_{\hat\gamma} = \hat\gamma^{-1}(\hat\alpha\hat\delta - 1) \in A_{\hat\gamma}^\eta. \tag{37}$$

Let us introduce the differential operator:

$$\mathrm{Op}(\beta) = \beta - \frac{s}{4}\delta\partial_l^* - \frac{s}{4}\alpha\partial_r^* + \frac{s^2}{16}\gamma\partial_l^*\partial_r^*. \tag{38}$$

It is easy to check that the determinant relation is satisfied

$$\mathrm{Op}(\alpha)\,\mathrm{Op}(\delta) - \mathrm{Op}(\gamma)\,\mathrm{Op}(\beta) = 1$$

where $\mathrm{Op}(\gamma)$ denotes the operator of multiplication by γ. The following lemma describes $\hat\beta_{\hat\gamma}$ in terms of $\mathrm{Op}(\beta)$.

Lemma 4.3. *Let $\hat{\beta}_{\hat{\gamma}} \in A^{\eta}_{\hat{\gamma}}$ be the affiliated element defined above. The set*

$$\{\mathcal{Q}(f)z_{\hat{\gamma}} : f \in \mathrm{C}_c^\infty(G)\}$$

is a core of $\hat{\beta}_{\hat{\gamma}}$ and for any $f \in \mathrm{C}_c^\infty(G)$ we have

$$\hat{\beta}_{\hat{\gamma}}\mathcal{Q}(f)z_{\hat{\gamma}} = \mathcal{Q}(\mathrm{Op}(\beta)f)z_{\hat{\gamma}}. \tag{39}$$

Moreover,

$$\{(1+\hat{\beta}^*_{\hat{\gamma}}\hat{\beta}_{\hat{\gamma}})\mathcal{Q}(f)z_{\hat{\gamma}} : f \in \mathrm{C}_c^\infty(G)\} \tag{40}$$

is dense in $A_{\hat{\gamma}}$.

Proof. Formula (39) follows from equation (37) and point 3 of Theorem 4.1. Consider the affiliated element $T = \hat{\alpha}\hat{\delta} - 1 \in A^{\eta}_{\hat{\gamma}}$. Let us check that T and $\hat{\alpha}^*\hat{\alpha} + \hat{\delta}^*\hat{\delta}$ strongly commute:

$$\begin{aligned}\exp(it(\hat{\alpha}^*\hat{\alpha}+\hat{\delta}^*\hat{\delta}))&T\exp(it(\hat{\alpha}^*\hat{\alpha}+\hat{\delta}^*\hat{\delta}))\\ &= \exp(it\hat{\alpha}^*\hat{\alpha})\,\hat{\alpha}\exp(-it\hat{\alpha}^*\hat{\alpha})\exp(it\hat{\delta}^*\hat{\delta})\,\hat{\delta}\exp(-it\hat{\delta}^*\hat{\delta}) - 1\\ &= \exp(its\hat{\gamma}^*\hat{\gamma})\,\hat{\alpha}\exp(-its\hat{\gamma}^*\hat{\gamma})\,\hat{\delta} - 1 = T.\end{aligned} \tag{41}$$

Using Theorem D.1 we define

$$T' = (1+T^*T)\exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta}) \in A^{\eta}_{\hat{\gamma}}. \tag{42}$$

The equality:

$$\exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta}) = \exp(-\hat{\alpha}\hat{\alpha}^* - \hat{\delta}\hat{\delta}^*)$$

implies that

$$\begin{aligned}T' = 2\exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta}) + \hat{\alpha}^*\hat{\alpha}\exp(-\hat{\alpha}^*\hat{\alpha})\hat{\delta}^*\hat{\delta}\exp(-\hat{\delta}^*\hat{\delta})\\ - \hat{\alpha}\exp(-\hat{\alpha}^*\hat{\alpha})\hat{\delta}\exp(-\hat{\delta}^*\hat{\delta}) - \hat{\alpha}^*\exp(-\hat{\alpha}\hat{\alpha}^*)\hat{\delta}^*\exp(-\hat{\delta}^*\hat{\delta}).\end{aligned} \tag{43}$$

All factors of the above sum belong to $\mathrm{M}(A_{\hat{\gamma}})$, hence $T' \in \mathrm{M}(A_{\hat{\gamma}})$. Note that

$$T'\,\mathrm{D}(T^*T) = \exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta})(1+T^*T)\,\mathrm{D}(T^*T) = \exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta})A_{\hat{\gamma}} \tag{44}$$

and the right hand side of (44) is dense in $A_{\hat{\gamma}}$. Using the boundedness of T' and the density of $\mathrm{D}(T^*T)$ in $A_{\hat{\gamma}}$ we conclude that the set $T'\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}$ is also dense:

$$\overline{T'\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}}^{\|\cdot\|} = A_{\hat{\gamma}}. \tag{45}$$

Let $a = \exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta})\mathcal{Q}(f)z_{\hat{\gamma}}$ for some $f \in \mathrm{C}_c^\infty(G)$. Using formula (101) from Appendix C one can check that there exists a sequence $f_n \in \mathrm{C}_c^\infty(G)$ such that

$$\begin{aligned}\lim_{n\to\infty}\partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}f_nz_\gamma &= \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}\exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta})fz_\gamma\\ \lim_{n\to\infty}\partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}(1+T^*T)f_nz_\gamma &= \partial_l^k\partial_l^{*k'}\partial_r^m\partial_r^{*m'}(1+T^*T)\exp(-\hat{\alpha}^*\hat{\alpha} - \hat{\delta}^*\hat{\delta})fz_\gamma\end{aligned} \tag{46}$$

for any $k, k', m, m' \leq 5$. Representing A on $L^2(G)$, we treat $T = \hat{\alpha}\hat{\delta} - 1$ in the above formula as an operator acting on $L^2(G)$. Let us also mention that the convergence is in the uniform topology on $\mathrm{C}_0(G)$. Using (46), Theorem A.1 and the closedness of $1+T^*T$ we see that

$$(1+T^*T)a = \lim_{n\to\infty}(1+T^*T)\mathcal{Q}(f_n)z_{\hat{\gamma}}. \tag{47}$$

Combining (45) and (47) we get

$$\overline{(1+T^*T)\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}}^{\|\cdot\|} = A_{\hat{\gamma}}.$$

Noting that $\hat{\beta}_{\hat{\gamma}} = \hat{\gamma}^{-1}T$ and using Lemma D.5 we get

$$\overline{(1+\hat{\beta}^*_{\hat{\gamma}}\hat{\beta}_{\hat{\gamma}})(1+|\hat{\gamma}|^{-2})^{-1}\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}}^{\|\cdot\|} = A_{\hat{\gamma}}. \tag{48}$$

Finally, the inclusion:

$$(1+|\hat{\gamma}|^{-2})^{-1}\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}} \subset \mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}$$

and equation (48) shows that

$$\overline{(1+\hat{\beta}^*_{\hat{\gamma}}\hat{\beta}_{\hat{\gamma}})\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat{\gamma}}}^{\|\cdot\|} = A_{\hat{\gamma}}.$$

This implies that $\mathcal{Q}(\mathrm{C}_c^\infty(G))z_{\hat\gamma}$ is a core of $\hat\beta_{\hat\gamma}$ (c.f. Lemma D.3) which ends the proof of our theorem. □

Using $\hat\beta_{\hat\gamma} \in A^\eta_{\hat\gamma}$ defined above and $\hat\beta_0 \in A^\eta_0$ defined in the previous section, we construct an affiliated element $\hat\beta \in A^\eta$. Heuristically speaking, it is a gluing of $\hat\beta_{\hat\gamma}$ and $\hat\beta_0$.

Theorem 4.4. *Let* $\mathrm{Op}(\beta)$ *be the differential operator* (38). *There exists an affiliated element* $\hat\beta \in A^\eta$ *such that the set* $\{\mathcal{Q}(f) : f \in \mathrm{C}_c^\infty(G)\}$ *is a core of* $\hat\beta$ *and*

$$\hat\beta\mathcal{Q}(f) = \mathcal{Q}(\mathrm{Op}(\beta)f) \tag{49}$$

for any $f \in \mathrm{C}_c^\infty(G)$.

Proof. Let $\mathrm{Graph}\,\hat\beta_{\hat\gamma} \in A^\eta_{\hat\gamma}$ be the graph of the affiliated element $\hat\beta_{\hat\gamma}$. It is easy to check that the set:

$$\left\{\begin{pmatrix} b \\ b'\end{pmatrix} : \begin{pmatrix} b\,a_{\hat\gamma} \\ b' a_{\hat\gamma}\end{pmatrix} \in \mathrm{Graph}(\hat\beta_{\hat\gamma}),\ \text{for any } a_{\hat\gamma} \in A_{\hat\gamma}\right\} \subset A \oplus A$$

is a graph of a closed operator acting on A. This operator will be denoted by $\hat\beta$. Let us list some properties of $\mathrm{Graph}\,\hat\beta$:

1. $\mathrm{Graph}\,\hat\beta \subset A \oplus A$ is a submodule of a Hilbert A-module $A \oplus A$;
2. For any $f \in \mathrm{C}_c^\infty(G)$ we have $\begin{pmatrix} \mathcal{Q}(f) \\ \mathcal{Q}(\mathrm{Op}(\beta)f)\end{pmatrix} \in \mathrm{Graph}\,\hat\beta$;
3. Let $(\mathrm{Graph}\,\hat\beta)^\perp$ be the submodule perpendicular to $\mathrm{Graph}\,\hat\beta$:
$$(\mathrm{Graph}\,\hat\beta)^\perp = \left\{\begin{pmatrix} c \\ c'\end{pmatrix} : c^*a + c'^*a' = 0 \text{ for any } \begin{pmatrix} a \\ a'\end{pmatrix} \in \mathrm{Graph}\,\hat\beta\right\}.$$
For any $f \in \mathrm{C}_c^\infty(G)$ we have $\begin{pmatrix} -\mathcal{Q}(\mathrm{Op}(\beta)^* f) \\ \mathcal{Q}(f)\end{pmatrix} \in (\mathrm{Graph}\,\hat\beta)^\perp$;
4. $\overline{\{\mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))f) : f \in \mathrm{C}_c^\infty(G)\}}^{\|\cdot\|} = A$.

Properties 1, 2 and 3 are consequnces of the definition of $\hat\beta$ and Lemma 4.3. In what follows we shall prove the last of the above properties:

$$\overline{\{\mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))f) : f \in \mathrm{C}_c^\infty(G)\}}^{\|\cdot\|} = A.$$

Let $a \in A$. Using (36) we can see that there exists a sequence $\tilde f_n \in \mathrm{C}_c^\infty(G_0)$ such that

$$\pi_0^\Psi(a) = \lim_{n\to\infty} \mathcal{Q}_0((1 + \mathrm{Op}(\beta_0)^*\,\mathrm{Op}(\beta_0))\tilde f_n). \tag{50}$$

Let $f_n \in \mathrm{C}_c^\infty(G)$ be an extension of $\tilde f_n$ to the whole group G and let π_0, π_0^Ψ be the morphisms introduced in Section 4.3. It is not difficult to check that

- $\pi_0^\Psi(\mathcal{Q}(f)) = \mathcal{Q}_0(\pi_0(f))$,
- $\pi_0(\mathrm{Op}(\beta)f) = \mathrm{Op}(\beta_0)\pi_0(f)$

Using these equalities and (50) we see that

$$\lim_{n\to\infty} \pi_0^\Psi(a - \mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))f_n)) = 0. \tag{51}$$

The exactness of sequnce (32) ensures that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $a_{\hat\gamma} \in A_{\hat\gamma}$ such that

$$\|a - \mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))f_n) - a_{\hat\gamma}\| \le \varepsilon. \tag{52}$$

Equality (40) implies that there exists a function $f \in \mathrm{C}_c^\infty(G)$ such that

$$\|a_{\hat\gamma} - \mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))f)z_{\hat\gamma}\| \le \varepsilon. \tag{53}$$

Combining (52) and (53) we get

$$\|a - \mathcal{Q}((1 + \mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))(f_n + f z_\gamma))\| \le 2\varepsilon.$$

This ends the proof of point 4 of our list.

Using the properties of Graph $\hat{\beta}$ one can check that it satisfies all the assumptions of Proposition 2.2 of [12]. This Proposition guaranties that $\hat{\beta} \in A^\eta$. It is easy to check that so constructed $\hat{\beta}$ satisfies all the requirements of our theorem.

□

4.5. **Representation theory of** C***-algebra** A**.** The results of Appendix B applied to the C*-algebra A of the Heisenberg-Lorentz quantum group $\mathbb{G}$ show that the representation theory of A can be equivalently described by the corepresentation theory of the dual quantum group $\hat{\mathbb{G}}$. For a given representation $\pi_U \in \mathrm{Rep}(A;\mathcal{H})$ the corresponding corepresentation $U_\pi \in \mathrm{M}(\mathcal{K}(H)\otimes\hat{A})$ is given by

$$U_\pi = (\pi_U \otimes \mathrm{id})\widehat{W} \tag{54}$$

where $\widehat{W} \in \mathrm{M}(A\otimes\hat{A})$ is the multiplicative unitary of $\hat{\mathbb{G}}$.

Theorem 4.5. *Let $(\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta})$ be a representation of the Heisenberg-Lorentz commutation relations on a Hilbert space $\mathcal{H}$ (c.f. Definition 3.3). There exists a unique representation $\pi \in \mathrm{Rep}(A;\mathcal{H})$ such that:*

$$\pi(\hat{\alpha}) = \tilde{\alpha}, \quad \pi(\hat{\beta}) = \tilde{\beta}, \quad \pi(\hat{\gamma}) = \tilde{\gamma}, \quad \pi(\hat{\delta}) = \tilde{\delta}. \tag{55}$$

Moreover, for any $\pi \in \mathrm{Rep}(A;\mathcal{H})$ the quadruple $(\pi(\hat{\alpha}),\pi(\hat{\beta}),\pi(\hat{\gamma}),\pi(\hat{\delta}))$ is a representation of the Heisenberg-Lorentz commutation relations.

Proof. We shall start by fixing some notation. Let $w \in \mathbb{C}\setminus\{0\}$ and $g_w \in G$ be given:

$$g_w = \begin{pmatrix} 0 & w^{-1} \\ -w & 0 \end{pmatrix}.$$

For any normal, invertible element n, acting on a Hilbert $\mathcal{H}$ we define the unitary operator $S(n)$:

$$S(n) = \int dE^n(w)\otimes R_{g_w} \in \mathrm{B}(\mathcal{H}\otimes L^2(G)) \tag{56}$$

where E^n is the spectral measure of n and $R_{g_w} \in \mathrm{B}(L^2(G))$ is the right shift by g_w. Let E^R be the spectral measure that corresponds to the representation $\Gamma \ni \gamma \mapsto R_\gamma \in \mathrm{B}(L^2(G))$ via the S.N.A.G. theorem. For any infinitesimal representation $(\tilde{a},\tilde{\lambda})$ of $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ we introduce the unitary operator $R(\tilde{a}) \in \mathrm{B}(\mathcal{H}\otimes L^2(G))$:

$$R(\tilde{a}) = \int U^{\tilde{a}}_{-z,0}\otimes dE^R(z). \tag{57}$$

Let us pass on to the main part of the proof. An immediate consequence of Definition 3.3 is that it is enough to consider two cases:

1. $\tilde{\gamma} = 0$;
2. $\ker\tilde{\gamma} = \{0\}$.

We give the proof for the second case leaving the first one to the reader. Using Theorem D.2 we define two closed operators $\tilde{\alpha}\tilde{\gamma}^{-1}, \tilde{\delta}\tilde{\gamma}^{-1}$ acting on $\mathcal{H}$. Note that $(\tilde{\alpha}\tilde{\gamma}^{-1},-s)$ and $(\tilde{\delta}\tilde{\gamma}^{-1},s)$ are infinitesimal representations of $\mathbb{H}$. Using the notation introduced above we define the unitary operator $U \in \mathrm{B}(\mathcal{H}\otimes L^2(G))$:

$$U = R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}) \tag{58}$$

which we will prove to be a corepresentation of $\hat{\mathbb{G}}$. Let $\hat{\Delta} \in \mathrm{Mor}(\mathrm{C}^*_r(G);\mathrm{C}^*_r(G)\otimes\mathrm{C}^*_r(G))$ be the canonical comultiplication on $\mathrm{C}^*_r(G)$. Note that

$$\begin{aligned}(\mathrm{id}\otimes\hat{\Delta})R(\tilde{\delta}\tilde{\gamma}^{-1}) &= \int U^{\tilde{\delta}\tilde{\gamma}^{-1}}_{-(z+z'),0}\otimes dE^R(z)\otimes dE^R(z') \\ &= \int \exp\left(-i\frac{s}{4}\Im(z\bar{z}')\right)U^{\tilde{\delta}\tilde{\gamma}^{-1}}_{-z,0}U^{\tilde{\delta}\tilde{\gamma}^{-1}}_{-z',0}\otimes dE^R(z)\otimes dE^R(z') \\ &= X^*_{23}R(\tilde{\delta}\tilde{\gamma}^{-1})_{12}R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}.\end{aligned} \tag{59}$$

The unitary element $X \in \mathrm{M}(\mathrm{C}^*_r(G)\otimes\mathrm{C}^*_r(G))$ used above:

$$X = \int \exp\left(i\frac{s}{4}\Im(z\bar{z}')\right)dE^R(z)\otimes dE^R(z') \tag{60}$$

is the one that twists $\hat{\Delta}$, giving comultiplication $\hat{\Delta}^{\Psi}$ of $\hat{\mathbb{G}}$ (see Theorem 4.12 of [2]):

$$\hat{\Delta}^{\Psi}(a) = X\hat{\Delta}(a)X^{*} \tag{61}$$

for any $a \in \mathrm{C}^{*}_{r}(G)$. Similarly we check that

$$(\mathrm{id}\otimes\hat{\Delta})R(\tilde{\alpha}\tilde{\gamma}^{-1}) = R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13}X_{23}. \tag{62}$$

Moreover, the formula $\hat{\Delta}(Z_w) = Z_w\otimes Z_w$ implies that:

$$(\mathrm{id}\otimes\hat{\Delta})S(\tilde{\gamma}) = S(\tilde{\gamma})_{12}S(\tilde{\gamma})_{13}. \tag{63}$$

Using equations (59), (62), (63), the fact that the first legs of $R(\tilde{\alpha}\tilde{\gamma}^{-1}), R(\tilde{\delta}\tilde{\gamma}^{-1})$ and $S(\tilde{\gamma})$ commute and formula (61) we get:

$$\begin{aligned}(\mathrm{id}\otimes\hat{\Delta}^{\Psi})U &= X_{23}(\mathrm{id}\otimes\hat{\Delta})(R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}))X_{23}^{*}\\ &= X_{23}X_{23}^{*}R(\tilde{\delta}\tilde{\gamma}^{-1})_{12}R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}S(\tilde{\gamma})_{12}S(\tilde{\gamma})_{13}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13}X_{23}X_{23}^{*}\\ &= R(\tilde{\delta}\tilde{\gamma}^{-1})_{12}S(\tilde{\gamma})_{12}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{12}R(\tilde{\delta}\tilde{\gamma}^{-1})_{13}S(\tilde{\gamma})_{13}R(\tilde{\alpha}\tilde{\gamma}^{-1})_{13} = U_{12}U_{13},\end{aligned}$$

which shows that U is a corepresentation of $\hat{\mathbb{G}}$. Let $\pi_U \in \mathrm{Rep}(A;\mathcal{H})$ be the corresponding representation of C^{*}-algebra A. The next step is to prove that π_U is the representation π of our theorem:

$$\pi_U(\hat{\alpha}) = \tilde{\alpha},\ \ \pi_U(\hat{\beta}) = \tilde{\beta},\ \ \pi_U(\hat{\gamma}) = \tilde{\gamma},\ \ \pi_U(\hat{\delta}) = \tilde{\delta}.$$

Treating $\hat{\alpha},\hat{\gamma},\hat{\delta}\in A^{\eta}$ as closed operators acting on $L^2(G)$ (in particular $\hat{\gamma}$ is an invertible operator of multiplication by the coordinate γ) one can prove that the multiplicative unitary $\widehat{W}$ is given by

$$\widehat{W} = R(\hat{\delta}\hat{\gamma}^{-1})S(\hat{\gamma})R(\hat{\alpha}\hat{\gamma}^{-1}). \tag{64}$$

It can also be proven that

$$\begin{aligned}\widehat{W}^{*}(1\otimes\exp(i\Im(z\hat{\gamma}))\widehat{W} &= U_{z,0}^{\hat{\alpha}\otimes\hat{\gamma}}U_{z,0}^{\hat{\gamma}\otimes\hat{\delta}},\\ U^{*}(1\otimes\exp(i\Im(z\hat{\gamma}))U &= U_{z,0}^{\tilde{\alpha}\otimes\hat{\gamma}}U_{z,0}^{\tilde{\gamma}\otimes\hat{\delta}},\end{aligned} \tag{65}$$

which implies that

$$\begin{aligned}\widehat{W}^{*}(1\otimes\hat{\gamma})\widehat{W} &= \hat{\alpha}\otimes\hat{\gamma}\dotplus\hat{\gamma}\otimes\hat{\delta},\\ U^{*}(1\otimes\hat{\gamma})U &= \tilde{\alpha}\otimes\hat{\gamma}\dotplus\tilde{\gamma}\otimes\hat{\delta}.\end{aligned} \tag{66}$$

Applying $\pi_U\otimes\mathrm{id}$ to both sides of the first of these equations and using (54) we get

$$\pi_U(\hat{\alpha})\otimes\hat{\gamma}\dotplus\pi_U(\hat{\gamma})\otimes\hat{\delta} = \tilde{\alpha}\otimes\hat{\gamma}\dotplus\tilde{\gamma}\otimes\hat{\delta}. \tag{67}$$

Let $\pi_0^{\Psi}\in\mathrm{Mor}(A;A_0)$ be the morphism introduced in Section 4.3. It sends $\hat{\gamma}$ to 0 and $\hat{\delta}$ to the normal element $\hat{\delta}_0 = \hat{\alpha}_0^{-1}\in A_0^{\eta}$. Applying $\mathrm{id}\otimes\pi_0^{\Psi}$ to both sides of (67) we get

$$\pi_U(\hat{\gamma})\otimes\hat{\delta}_0 = \tilde{\gamma}\otimes\hat{\delta}_0. \tag{68}$$

This immediately implies that $\pi_U(\hat{\gamma}) = \tilde{\gamma}$. From this equality and (67) we see that $\pi_U(\hat{\alpha}) = \tilde{\alpha}$. Now using (54) and (64) we get

$$R(\pi_U(\hat{\delta})\tilde{\gamma}^{-1}) = R(\tilde{\delta}\tilde{\gamma}^{-1}).$$

The equation (57) together with the fact that the support of the measure dE^{R} is the whole complex plain implies that $\pi_U(\hat{\delta})\tilde{\gamma}^{-1} = \tilde{\delta}\tilde{\gamma}^{-1}$, hence $\pi_U(\hat{\delta}) = \tilde{\delta}$. Finally, $\pi_U(\hat{\beta}) = \tilde{\beta}$ which is a consequence of the related equalities for $\hat{\alpha},\hat{\gamma}$ and $\hat{\delta}$.

The fact that for any representation $\pi\in\mathrm{Rep}(A;\mathcal{H})$ the quadruple $(\pi(\hat{\alpha}),\pi(\hat{\beta}),\pi(\hat{\gamma}),\pi(\hat{\delta}))$ is a representation of Heisenberg-Lorentz commutation relations follows directly from the definitions of affiliated elements $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\delta}\in A^{\eta}$. $\square$

The above theorem implies the following

Corollary 4.6. *Let A be the* C^**-algebra of the Heisenberg-Lorentz quantum group. Then the generators $\hat\alpha,\hat\beta,\hat\gamma,\hat\delta\in A^\eta$ separate representations of A, i.e. if π_1 and $\pi_2\in\mathrm{Rep}(A;\mathcal{H})$ coincide on $\hat\alpha,\hat\beta,\hat\gamma,\hat\delta$:*

$$\pi_1(\hat\alpha)=\pi_2(\hat\alpha)\,,\ \pi_1(\hat\beta)=\pi_2(\hat\beta)\,,\ \pi_1(\hat\gamma)=\pi_2(\hat\gamma)\,,\ \pi_1(\hat\delta)=\pi_2(\hat\delta)$$

then $\pi_1=\pi_2$.

4.6. $\hat\alpha,\hat\beta,\hat\gamma,\hat\delta$ **generate** A. By Corollary 4.6 we know that affiliated elements $\hat\alpha,\hat\beta,\hat\gamma,\hat\delta\in A^\eta$ separate representations of A. The aim of this section is to prove that they generate A in the sense of Woronowicz. For this purpose we shall use the following theorem which is a consequence of Theorem 4.2 of [11].

Theorem 4.7. *Let $T_1,T_2\dots,T_n$ be elements affiliated with a* C^**-algebra A. Let Ω be the subset of $M(A)$ consisting of elements of the form $(1+T_i^*T_i)^{-1}$, $(1+T_iT_i^*)^{-1}$, $\exp(-T_i^*T_i)$, $\exp(-T_iT_i^*)$. Assume that*

1. *$T_1,T_2\dots,T_n$ separate representations;*
2. *there exist elements $r_1,r_2,\dots,r_k\in\Omega$ such that $r_1r_2\dots r_k\in A$.*

Then $T_1,T_2,\dots,T_n$ generate A.

Theorem 4.8. *Affiliated elements $\hat\alpha,\hat\beta,\hat\gamma,\hat\delta$ generate the* C^**-algebra A of the Heisenberg-Lorentz quantum group.*

Proof. From Theorem 4.7 and Corollary 4.6 we see that it is enough to prove that

$$(1+\hat\beta^*\hat\beta)^{-1}\exp(-\hat\alpha^*\hat\alpha)\exp(-\hat\delta^*\hat\delta),$$

which is an element of $\mathrm{M}(A)$, belongs in fact to $A\subset\mathrm{M}(A)$. For any $g\in G$ we set

$$f(g)=\frac{1}{\cosh^2\left(\frac{s}{2}\gamma^*\gamma\right)}\exp\left(-\frac{2(|\alpha|^2+|\delta|^2)\tanh\left(\frac{s}{2}\gamma^*\gamma\right)}{s\gamma^*\gamma}\right).$$

Let h_t be the family of functions defined by (100). A straightforward computation shows that

$$f(g)=\int d^2z_1d^2z_2\,h_1\left(z_1,-\frac{s}{2}|\gamma|^2\right)h_1\left(z_2,\frac{s}{2}|\gamma|^2\right)\exp(i\Im(z_1\alpha))\exp(i\Im(z_2\delta)).\tag{69}$$

One can check that the functions $\exp(-|\gamma|^2)\exp(i\Im(z_1\alpha))$ and $\exp(-|\gamma|^2)\exp(i\Im(z_2\delta))$ are quantizable in the sense of Theorem A.4 and

$$\begin{aligned}\mathcal{Q}(\exp(-|\gamma|^2)\exp(i\Im(z_1\alpha)))&=\exp(-|\gamma|^2)U^{\hat\alpha}_{z,0},\\ \mathcal{Q}(\exp(-|\gamma|^2)\exp(i\Im(z_2\delta)))&=\exp(-|\gamma|^2)U^{\hat\delta}_{z,0}.\end{aligned}\tag{70}$$

Using (69), (70) and (101) we get

$$\mathcal{Q}(f)=\exp(-\hat\alpha^*\hat\alpha)\exp(-\hat\delta^*\hat\delta).\tag{71}$$

Let us define two auxiliary functions $k_1,k_2:G\to\mathbb{C}$:

$$\begin{aligned}k_1(g)&=\frac{1}{1+\bar\beta\beta}f(g),\\ k_2(g)&=f-(1+\mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))k_1.\end{aligned}\tag{72}$$

They satisfy the assumptions of Theorem A.1 hence we can quantize them obtaining $\mathcal{Q}(k_1),\mathcal{Q}(k_2)\in A$. Combining (71) and (72) we see that

$$\begin{aligned}(1+\hat\beta^*\hat\beta)^{-1}\exp(-\hat\alpha^*\hat\alpha)\exp(-\hat\delta^*\hat\delta)&=(1+\hat\beta^*\hat\beta)^{-1}(\mathcal{Q}(f))\\ &=(1+\hat\beta^*\hat\beta)^{-1}\mathcal{Q}((1+\mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))k_1+k_2)\\ &=(1+\hat\beta^*\hat\beta)^{-1}\mathcal{Q}((1+\mathrm{Op}(\beta)^*\,\mathrm{Op}(\beta))k_1)+(1+\hat\beta^*\hat\beta)^{-1}\mathcal{Q}(k_2)\\ &=\mathcal{Q}(k_1)+(1+\hat\beta^*\hat\beta)^{-1}\mathcal{Q}(k_2).\end{aligned}$$

Both factors of the above sum belong to A, therefore $(1+\hat\beta^*\hat\beta)^{-1}\exp(-\hat\alpha^*\hat\alpha)\exp(-\hat\delta^*\hat\delta)\in A$. $\square$

4.7. **Comultiplication.**

Theorem 4.9. *Let $\mathbb{G} = (A, \Delta)$ be the Heisenberg-Lorentz quantum group. Then the action of Δ on the generators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta} \in A^\eta$ is given by*

$$\begin{aligned} \Delta(\hat{\alpha}) &= \hat{\alpha} \otimes \hat{\alpha} \dotplus \hat{\beta} \otimes \hat{\gamma} \\ \Delta(\hat{\beta}) &= \hat{\alpha} \otimes \hat{\beta} \dotplus \hat{\beta} \otimes \hat{\delta} \\ \Delta(\hat{\gamma}) &= \hat{\gamma} \otimes \hat{\alpha} \dotplus \hat{\delta} \otimes \hat{\gamma} \\ \Delta(\hat{\delta}) &= \hat{\gamma} \otimes \hat{\beta} \dotplus \hat{\delta} \otimes \hat{\delta}. \end{aligned} \tag{73}$$

Remark 4.10. The affiliated elements appearing on the right hand side of the formula for comultiplication (73) are treated as closed operators acting on $L^2(G) \otimes L^2(G)$ and the sign $\dotplus$ denotes the closure of the sum of two operators (see also Theorem 3.5).

Proof. In this proof we shall use the notation of the proof of Theorem 4.5. Let $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ denote the right hand sides of (73). By Theorem 3.5 $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})$ is a representation of the Heisenberg-Lorentz commutation relations on $L^2(G) \otimes L^2(G)$ with invertible operator $\tilde{\gamma}$. Using the fact that

$$(\Delta \otimes \mathrm{id})\widehat{W} = \widehat{W}_{23}\widehat{W}_{13}$$

and correspondence (54) we conclude that our theorem can be deduced from the equality

$$\widehat{W}_{23}\widehat{W}_{13} = R(\tilde{\delta}\tilde{\gamma}^{-1})S(\tilde{\gamma})R(\tilde{\alpha}\tilde{\gamma}^{-1}), \tag{74}$$

which we prove below. From equation (64) we can see that:

$$\begin{aligned} \widehat{W}_{13} &= R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1), \\ \widehat{W}_{23} &= R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1}). \end{aligned} \tag{75}$$

Therefore, the right hand side of (74) has the following form:

$$R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1). \tag{76}$$

Using the fact that $R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})$ commutes with $R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)$ we can see that (76) is equal to

$$R(1 \otimes \hat{\delta}\hat{\gamma}^{-1})S(1 \otimes \hat{\gamma})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})S(\hat{\gamma} \otimes 1)R(\hat{\alpha}\hat{\gamma}^{-1} \otimes 1). \tag{77}$$

Formula (12) and the analogous formula related to $\tilde{\delta}$ imply that

$$\begin{aligned} \hat{\alpha}\hat{\gamma}^{-1} \otimes 1 &= \tilde{\alpha}\tilde{\gamma}^{-1} \dotplus \tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}), \\ 1 \otimes \hat{\delta}\hat{\gamma}^{-1} &= \tilde{\delta}\tilde{\gamma}^{-1} \dotplus \tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}). \end{aligned} \tag{78}$$

Using these equalities and the fact that (76) is equal to $\widehat{W}_{23}\widehat{W}_{13}$ we get:

$$\begin{aligned} \widehat{W}_{23}\widehat{W}_{13} = {} & R(\tilde{\delta}\tilde{\gamma}^{-1}) \exp(-i\Im(\tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}) \otimes T_r))S(1 \otimes \hat{\gamma})R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1) \\ & \times R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1})S(\hat{\gamma} \otimes 1) \exp(-i\Im(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}) \otimes T_r))R(\tilde{\alpha}\tilde{\gamma}^{-1}). \end{aligned} \tag{79}$$

Noting that

$$R(\hat{\delta}\hat{\gamma}^{-1} \otimes 1)R(1 \otimes \hat{\alpha}\hat{\gamma}^{-1}) = \exp(-i\Im(\tilde{\gamma}(\hat{\gamma}^{-1} \otimes \hat{\gamma}^{-1}) \otimes T_r))$$

and using equation (79) we see that to in order to prove equality (74) it is enough to check that

$$\begin{aligned} S(\tilde{\gamma}) = {} & \exp(-i\Im(\tilde{\gamma}^{-1}(\hat{\gamma} \otimes \hat{\gamma}^{-1}) \otimes T_r))S(1 \otimes \hat{\gamma}) \\ & \times \exp(-i\Im(\tilde{\gamma}(\hat{\gamma}^{-1} \otimes \hat{\gamma}^{-1}) \otimes T_r))S(\hat{\gamma} \otimes 1) \exp(-i\Im(\tilde{\gamma}^{-1}(\hat{\gamma}^{-1} \otimes \hat{\gamma}) \otimes T_r)). \end{aligned} \tag{80}$$

Operators $\tilde{\gamma}, 1 \otimes \hat{\gamma}, \hat{\gamma} \otimes 1$ which appear in the above expression are normal and they strongly commute. Therefore, to prove (80) it is enough to check that

$$S(u) = \exp(-i\Im(u^{-1}vw^{-1}T_r))S(w) \exp(-i\Im(uv^{-1}w^{-1}T_r))S(v) \exp(-i\Im(u^{-1}v^{-1}wT_r)) \tag{81}$$

for any $u, v, w \in \mathbb{C} \setminus \{0\}$. Noting that

$$S(w) = Z_w, \;\; \exp(i\Im(zT_r)) = R_z, \tag{82}$$

where Z_w and R_z were defined in the proof of Theorem 4.5, we see that equation (81) is equivalent to the following matrix identity:

$$\begin{pmatrix} 0 & u^{-1} \\ -u & 0 \end{pmatrix} = \begin{pmatrix} 1 & -vu^{-1}w^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & w^{-1} \\ -w & 0 \end{pmatrix} \begin{pmatrix} 1 & -uv^{-1}w^{-1} \\ 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & v^{-1} \\ -v & 0 \end{pmatrix} \begin{pmatrix} 1 & -wu^{-1}v^{-1} \\ 0 & 1 \end{pmatrix}.$$

Its verification is a straightforward computation which is left to the reader. □

Appendices

A. Quantization map

Let $\mathcal{Q}$ be the quantization map introduced in Definition 4.13 of paper [2]. Recall that $\mathcal{Q}$ was defined on elements of the Fourier algebra $\mathcal{F}$:

$$\mathcal{F} = \{(\omega \otimes \mathrm{id})V : \omega \in \mathrm{B}(L^2(G))_*\}$$

and $\mathcal{Q}((\omega \otimes \mathrm{id})V) = (\omega \otimes \mathrm{id})W$, where V is the Kac-Takesaki operator of a locally compact group G and W is the multiplicative unitary related to a Rieffel deformation of G. Given a function $f \in \mathrm{C}_0(G)$ it is usually difficult to check if $f \in \mathcal{F}$, which makes $\mathcal{Q}$ not very useful. In the case of the Heisenberg-Lorentz quantum group we shall give a new description of the quantization map which does not posses the aforementioned drawback. $\mathcal{Q}$ will be defined on a different class of functions but when the function happens to be an element of $\mathcal{F}$ then the new definition will coincide with the old one. In fact our construction works in the case of the Rieffel deformation with $\Gamma \cong \mathbb{C}$. Consider two representations of $\Gamma \subset G$ on $L^2(G)$:

$$\begin{aligned} \Gamma \ni \gamma \mapsto R_\gamma \in \mathrm{B}(L^2(G)), \\ \Gamma \ni \gamma \mapsto L_\gamma \in \mathrm{B}(L^2(G)). \end{aligned} \tag{83}$$

Let T_l and T_r be infinitesimal generators of these representations:

$$\begin{aligned} R_\gamma &= \exp(i\Im(\gamma T_r)), \\ L_\gamma &= \exp(i\Im(\gamma T_l)) \end{aligned} \tag{84}$$

for any $\gamma \in \Gamma$. The related complex vector fields on G will be denoted by ∂_l and ∂_r (see equation (24)). Now consider two differential operators $K_l = (1 + T_l^* T_l)^2$ and $K_r = (1 + T_r^* T_r)^2$ acting on $L^2(G)$. Note that K_l and K_r are positive, invertible and their inverses K_l^{-1}, K_r^{-1} are bounded.

Let $x, y, v, w \in L^2(G)$ and assume that $x, y, w \in \mathrm{D}(K_r)$ and $v \in \mathrm{D}(K_l)$. Our next objective is to compute the matrix element $\langle\, x \otimes v \,|\, W \,|\, y \otimes w \,\rangle$. Note that

$$\langle\, x \otimes v \,|\, W \,|\, y \otimes w \,\rangle = \langle\, K_r x \otimes K_l v \,|\, (K_r^{-1} \otimes K_l^{-1}) Y V X (K_r^{-1} \otimes K_r^{-1}) \,|\, K_r y \otimes K_r w \,\rangle. \tag{85}$$

Let $\pi_R, \pi_L \in \mathrm{Rep}(\mathrm{C}_0(\mathbb{C}); L^2(G))$ be representations of $\mathrm{C}_0(\mathbb{C})$ which send $\mathrm{id} \in \mathrm{C}_0(\mathbb{C})^\eta$ to T_r and T_l respectively. We have the following two equalities:

$$X(K_r^{-1} \otimes K_r^{-1}) = (\pi_R \otimes \pi_R)((K^{-1} \otimes K^{-1})\Psi) \tag{86}$$

$$(K_r^{-1} \otimes K_l^{-1})Y = (\pi_R \otimes \pi_L)((K^{-1} \otimes K^{-1})\Psi), \tag{87}$$

where $K : \mathbb{C} \to \mathbb{R}$ is the function given by the formula:

$$K(z) = (1 + |z|^2)^2$$

and $\Psi \in \mathrm{M}(\mathrm{C}_0(\mathbb{C}) \otimes \mathrm{C}_0(\mathbb{C}))$ is defined by (15). Let $l : \mathbb{C}^2 \to \mathbb{C}$ be the function given by:

$$l(w_1, w_2) = \int d^2 z_1 d^2 z_2 \, \frac{\exp(-i\Im(z_1 w_1 + z_2 w_2 - \frac{s}{4} w_1 \bar{w}_2))}{(1 + |z_1|^2)^2 (1 + |z_2|^2)^2}.$$

Note that $l \in L^1(\mathbb{C}^2)$ and the right hand side of (86) can be expressed by l:

$$(\pi_R \otimes \pi_R)((K^{-1} \otimes K^{-1})\Psi) = \int d^2 w_1 d^2 w_2 \, l(w_1, w_2)(R_{w_1} \otimes R_{w_2}). \tag{88}$$

We have a similar formula for the right hand side of (87):

$$(\pi_R \otimes \pi_L)((K^{-1} \otimes K^{-1})\Psi) = \int d^2w_1 d^2w_2\, l(w_1, w_2)(R_{w_1} \otimes L_{w_2}). \tag{89}$$

Let $f = (\omega_{x,y} \otimes \mathrm{id})V \in \mathrm{C}_0(G)$ be the slice of the Kac-Takesaki operator and $\tilde{h} \in \mathrm{C}_0(\mathbb{C})$ the function given by:

$$\tilde{h}(w) = \int dz \frac{\exp(i\Im(wz))}{(1+s^{-2}|z|^2)^2}. \tag{90}$$

Let $\pi^{\mathrm{can}} \in \mathrm{Rep}(\mathrm{C}_0(G) \rtimes \mathbb{C}^2; L^2(G))$ be the representation introduced in Remark 4.5 of [2] and $\lambda^L, \lambda^R \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{C}); \mathrm{C}_0(G) \rtimes \mathbb{C}^2)$ the morphisms introduced in the paragraph following Proposition 4.2 of [2]. A simple but tedious computation which starts with entering (88) and (89) into (85) leads to the following equality:

$$\begin{aligned}&\langle\, x \otimes v \,|\, W \,|\, y \otimes w \,\rangle \\ &\quad = \int d^2w_1 d^2w_2 \langle\, v \,|\, \pi^{\mathrm{can}}\Big(\hat{\rho}^{\Psi}_{w_1,w_2}\big(\lambda^L(\tilde{h})\big((1+\partial_r^*\partial_r)^2(1+\partial_l^*\partial_l)^2 f\big)\lambda^R(\tilde{h})\big)\Big) \,|\, w \,\rangle\end{aligned}$$

Denoting

$$\lambda^L(\tilde{h})\big((1+\partial_r^*\partial_r)^2(1+\partial_l^*\partial_l)^2 f\big)\lambda^R(\tilde{h}) \in \mathrm{M}(\mathrm{C}_0(G) \rtimes \mathbb{C}^2)$$

by b_f we get

$$\langle\, x \otimes v \,|\, W \,|\, y \otimes w \,\rangle = \int d^2w_1 d^2w_2 \langle\, v \,|\, \pi^{\mathrm{can}}(\hat{\rho}^{\Psi}_{w_1,w_2}(b_f)) \,|\, w \,\rangle. \tag{91}$$

If b_f happens to be in the domain of $\mathfrak{E}^{\Psi}$ - the averaging map with respect to the twisted dual action $\hat{\rho}^{\Psi}$ - then, in the formula (91) we can enter the integral under the scalar product obtaining

$$\langle\, x \otimes v \,|\, W \,|\, y \otimes w \,\rangle = \langle\, v \,|\, \pi^{\mathrm{can}}\left(\int d^2w_1 d^2w_2\, \hat{\rho}^{\Psi}_{w_1,w_2}(b_f)\right) \,|\, w \,\rangle. \tag{92}$$

Equation (92) may then be rewritten as follows:

$$\mathcal{Q}(f) = \pi^{\mathrm{can}}(\mathfrak{E}^{\Psi}(b_f)).$$

Let us show that this last equation holds whenever f is regular enough. In the next theorem we shall keep the same notation T_l and T_r for normal operators acting on $L^2(G)$ (see (84)) and elements affiliated with $\mathrm{C}_0(G) \rtimes \mathbb{C}^2$ (see (22)).

Theorem A.1. *Let ∂_l, ∂_r be the complex vector fields on G given by* (24) *and $f \in \mathrm{C}_0(G)$ a continuous function such that $\partial_l^{k*}\partial_l^{k'}\partial_r^{*m}\partial_r^{m'} f \in \mathrm{C}_0(G)$ whenever $k, k', l, l' \leq 5$. Let $b_f \in \mathrm{M}(\mathrm{C}_0(G) \rtimes \mathbb{C}^2)$ be the element given by*

$$b_f = \lambda^L(\tilde{h})\big((1+\partial_r^*\partial_r)^2(1+\partial_l^*\partial_l)^2 f\big)\lambda^R(\tilde{h}).$$

Then b_f is in the domain of the averaging map $\mathfrak{E}^{\Psi}$ and there exists a positive constant c such that

$$\|\mathfrak{E}^{\Psi}(b_f)\| \leq c \max_{k,k',l,l' \leq 5} \sup_{g \in G} \left|\partial_l^{k*}\partial_l^{k'}\partial_r^{*m}\partial_r^{m'} f\right|.$$

If $f \in \mathrm{C}_0(G)$ is quantizable in the sense of Definition 4.13 of [2]*, then*

$$\mathcal{Q}(f) = \pi^{\mathrm{can}}(\mathfrak{E}^{\Psi}(f)).$$

To prove the above theorem we shall need the following lemma:

Lemma A.2. *Let X be a locally compact Hausdorff space, $\rho : \mathbb{C} \to \mathrm{Aut}(\mathrm{C}_0(X))$ a continuous action and $(\mathrm{C}_0(X) \rtimes \mathbb{C}, \lambda, \hat{\rho})$ the canonical $\mathbb{C}$-product associated with ρ. Let ∂, ∂^* be the differential operators acting on the smooth domain $\mathrm{D}^{\infty}(\rho) \subset \mathrm{C}_0(X)$ of the action ρ:*

$$\begin{aligned}\partial f &= \tfrac{\partial}{\partial z}\rho_z f|_{z=0} \\ \partial^* f &= \tfrac{\partial}{\partial \bar{z}}\rho_z f|_{z=0}\end{aligned}. \tag{93}$$

Further, let $\tilde{h} \in \mathrm{C}_0(\mathbb{C})$ be the function defined by (90) *and let $g \in \mathrm{C}_0(X)$ be such that $\partial^{*l}\partial^k g \in \mathrm{C}_0(\mathbb{C})$ for $k, l \in \{0, 1\}$. Then $\lambda(\tilde{h})g$ is in the domain of the averaging map $\mathfrak{E}$ and there exists a positive constant $c \in \mathbb{R}$ such that:*

$$\|\mathfrak{E}(\lambda(\tilde{h})g)\| \leq c \max_{l,k\leq 1} \sup_{x\in X} |\partial^{*l}\partial^k g(x)|. \tag{94}$$

Proof. By universal properties of the group C*-algebra $\mathrm{C}^*(\mathbb{C})$ $\lambda \in \mathrm{Rep}(\mathbb{C}; \mathrm{C}_0(X) \rtimes \mathbb{C})$ corresponds to a unique element of $\mathrm{Mor}(\mathrm{C}^*(\mathbb{C}); \mathrm{C}_0(X) \rtimes \mathbb{C})$, which will also be denoted by λ. Identifying $\mathrm{C}^*(\mathbb{C}) \cong \mathrm{C}_0(\mathbb{C})$ we can apply λ to $\tilde{h} \in \mathrm{C}_0(\mathbb{C})$: $\lambda(\tilde{h}) \in \mathrm{M}(\mathrm{C}_0(X) \rtimes \mathbb{C})$.

In order to show that $\lambda(\tilde{h})g$ is in the domain of the averaging map $\mathrm{D}(\mathfrak{E})$ it is enough to express it as a linear combination of elements of the form

$$\lambda(h_1)b\lambda(h_2) \tag{95}$$

where $h_1, h_2 \in \mathrm{C}_0(\mathbb{C}) \cap L^2(\mathbb{C})$ and $b \in \mathrm{C}_0(X) \rtimes \mathbb{C}$ (see [5]). Let $T \,\eta\, \mathrm{C}_0(X) \rtimes \mathbb{C}$ be the image of $\mathrm{id} \in \mathrm{C}_0(\mathbb{C})$ under $\lambda \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{C}); \mathrm{C}_0(X) \rtimes \mathbb{C})$: $T = \lambda(\mathrm{id})$. Note that

$$\begin{aligned}\lambda(\tilde{h})g &= \lambda(\tilde{h})g(1 + T^*T)(1 + T^*T)^{-1} \\ &= \lambda(\tilde{h})g(1 + T^*T)^{-1} + \lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1} + \lambda(\tilde{h})\partial^* gT(1 + T^*T)^{-1}\end{aligned} \tag{96}$$

where we used the relation linking ∂^* and T^*:

$$\partial^* g = [g, T^*].$$

Note also that $(1 + T^*T)^{-1} = \lambda((1 + |z|^2)^{-1})$, $(1 + |z|^2)^{-1} \in L^2(\mathbb{C})$ and

$$\lambda(\tilde{h}) = \lambda((1 + |z|^2)\tilde{h})(1 + T^*T)^{-1}.$$

Therefore, the first summand of the right hand side of (96) is of the form

$$\lambda(\tilde{h})g(1 + T^*T)^{-1} = \lambda((1 + |z|^2)\tilde{h})((1 + T^*T)^{-1}g)\lambda((1 + |z|^2)^{-1}).$$

Using the fact that $\tilde{h}$ is of the Schwartz type we can see that the above element is of the form (95). Now, by inequality (10) of paper [2] we get

$$\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\| \leq \|\tilde{h}\|_2 \, \|g\| \, \|(1 + |z|^2)^{-1}\|_2$$

and see that $\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\|$ may be estimated by the right hand side of (94) for c' big enough:

$$\|\mathfrak{E}(\lambda(\tilde{h})g(1 + T^*T)^{-1})\| \leq c' \max_{l,k\leq 1} \sup_{x\in X} |\partial^{*l}\partial^k g(x)| \tag{97}$$

Let us analyze the second summand of the right hand side of (96). Note that

$$\lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1} = \lambda(\tilde{h}|z|^2)g(1 + T^*T)^{-1} + \lambda(\tilde{h}\bar{z})\partial g(1 + T^*T)^{-1}.$$

A reasoning similar to the one above shows that there exists a constant c'' such that

$$\|\mathfrak{E}(\lambda(\tilde{h}\bar{z})gT(1 + T^*T)^{-1})\| \leq c'' \max_{l,k\leq 1} \sup_{x\in X} |\partial^{*l}\partial^k g(x)|. \tag{98}$$

Similarly, we prove that there exists a constant c''' such that

$$\|\mathfrak{E}(\lambda(\tilde{h})\partial^* gT(1 + T^*T)^{-1})\| \leq c''' \max_{l,k\leq 1} \sup_{x\in X} |\partial^{*l}\partial^k g(x)|. \tag{99}$$

Combining (96), (97), (98) and (99) we get (94) for $c = \max\{c', c'', c'''\}$. □

Remark A.3. The above lemma is also true if we replace $\mathfrak{E}$ with $\mathfrak{E}^\Psi$. An extension of this lemma to the case of an action of $\mathbb{C}^2$ leads to the proof of the Theorem A.1.

Using the techniques of the proof of Lemma A.2 one can also prove the following theorem:

Theorem A.4. *Let $f \in \mathrm{C}_{\mathrm{bounded}}(G)$ be a function such that $\partial_l^{k*}\partial_l^{k'}\partial_r^{*m}\partial_r^{m'} f \in \mathrm{C}_{\mathrm{bounded}}(G)$ whenever $k, k', l, l' \leq 5$. Let $b_f \in \mathrm{M}(\mathrm{C}_0(G) \rtimes \mathbb{C}^2)$ be given by:*

$$b_f = \lambda^L(\tilde{h})\big((1+\partial_r^*\partial_r)^2(1+\partial_l^*\partial_l)^2 f\big)\lambda^R(\tilde{h}).$$

Then $b_f \in \mathrm{D}(\mathfrak{E}^\Psi)$, $\mathfrak{E}^\Psi(b_f) \in \mathrm{M}(A)$ and there exists a positive constant c such that

$$\|\mathfrak{E}^\Psi(b_f)\| \leq c \max_{k,k',l,l' \leq 5} \sup_{g \in G} \left|\partial_l^{k*}\partial_l^{k'}\partial_r^{*m}\partial_r^{m'} f\right|.$$

The element $\mathfrak{E}^\Psi(b_f) \in \mathrm{M}(A)$ appearing in the above theorem will also be denoted by $\mathcal{Q}(f)$.

B. Counit in Rieffel deformation

The aim of this section is to show that a quantum group $\mathbb{G}$ obtained by the Rieffel deformation posseses a counit. Our argument will be different from the one given by Rieffel in [8]. Let G be a locally compact group, $\Gamma \subset G$ its abelian subgroup and Ψ a 2-cocycle on $\hat{\Gamma}$. Let ρ be the action of Γ^2 on $\mathrm{C}_0(G)$ given by the left and right shifts and ρ_Γ corresponding action of Γ^2 on $\mathrm{C}_0(\Gamma)$. Note that the restriction morphism $\pi_\Gamma : \mathrm{C}_0(G) \to \mathrm{C}_0(\Gamma)$ is Γ^2-covariant. Using Proposition 3.8 of [2] we get the induced morphism $\pi_\Gamma^\Psi \in \mathrm{Mor}(\mathrm{C}_0(G)^\Psi; \mathrm{C}_0(\Gamma)^\Psi)$. The abelianity of Γ implies that the dual quantum group of $(\mathrm{C}_0(\Gamma)^\Psi, \Delta)$ is just the standard quantum group $(\mathrm{C}^*(\Gamma), \hat{\Delta})$. Therefore $(\mathrm{C}_0(\Gamma)^\Psi, \Delta)$ coincides with $(\mathrm{C}_0(\Gamma), \Delta)$. This shows that $\pi_\Gamma^\Psi \in \mathrm{Mor}(\mathrm{C}_0(G)^\Psi; \mathrm{C}_0(\Gamma))$ and enables us to define a counit e for $\mathbb{G}$ by the formula $e(a) = e_\Gamma(\pi_\Gamma^\Psi(a))$ for any $a \in A$, where $e_\Gamma : \mathrm{C}_0(\Gamma) \to \mathbb{C}$ is the counit for $(\mathrm{C}_0(\Gamma), \Delta)$.

Let us now draw an important conclusion from the existence of the counit for $\mathbb{G}$. By Proposition 5.16 of [9] we see that the universal dual quantum group of $\hat{\mathbb{G}} = (\mathrm{C}_r^*(G), \Delta^\Psi)$ is isomorphic with the reduced dual: $\mathbb{G} = (A, \Delta)$. In particular, representations of C^*-algebra A are in one to one correspondence with representations of quantum group $\hat{\mathbb{G}}$. This follows from Theorem 5.4 of [9].

C. Complex generator of Heisenberg Lie algebra

Let $\mathbb{H}$ be the Heisenberg group, $\mathfrak{h}$ its Lie algebra and $\mathcal{E}$ the enveloping algebra of $\mathfrak{h}$. $\mathcal{E}$ is generated by an element $a \in \mathcal{E}$ such that the commutator $\lambda = [a^*, a]$ is central in $\mathcal{E}$. Let A be a C^*-algebra and let $U \in \mathrm{Rep}(\mathbb{H}; A)$ be a representation. As was described in the third chapter of [10], U induces the map

$$dU : \mathcal{E} \to \{\text{closed maps on } A\}.$$

By $\mathrm{D}^\infty(U)$ we shall denote the set of U-smooth elements in A. In the next definition we shall identify a representation of $\mathbb{H}$ in the C^*-algebra of compact operators $\mathcal{K}(\mathcal{H})$ with the corresponding Hilbert space representation.

Definition C.1. Let $\mathcal{H}$ be a Hilbert space and let $(\tilde{a}, \tilde{\lambda})$ be a pair of closed operators acting on $\mathcal{H}$. We say that this pair is an infinitesimal representation of $\mathbb{H}$ on $\mathcal{H}$ if there exists a representation $U \in \mathrm{Rep}(\mathbb{H}; \mathcal{K}(\mathcal{H}))$ such that $dU(a) = \tilde{a}$ and $dU(\lambda) = \tilde{\lambda}$.

The representation U in the above definition is determined by $\tilde{a}$, therefore in this context U will be denoted by $U^{\tilde{a}}$. Let $U \in \mathrm{Rep}(\mathbb{H}; \mathrm{C}^*(\mathbb{H}))$ be the canonical representation of $\mathbb{H}$. The map dU is in this case injective, which enables us to identify $dU(T)$ with $T \in \mathcal{E}$. The aim of this section is to show that $a \in \mathcal{E}$ is affiliated with $\mathrm{C}^*(\mathbb{H})$. In fact one can prove that a generates $\mathrm{C}^*(\mathbb{H})$ in the sense of Woronowicz but we shall not use and so not prove this fact. Let $M \in \mathcal{E}$. The criterion for a map $dU(M) : \mathrm{D}(dU(M)) \to \mathrm{C}^*(\mathbb{H})$ to be affiliated with $\mathrm{C}^*(\mathbb{H})$ is provided by Theorem 2.1 of [10]. Our proof that $a \,\eta\, \mathrm{C}^*(\mathbb{H})$ uses a different technique which is based on the explicite construction of the semigroup $\mathbb{R}_+ \ni t \mapsto \exp(-ta^*a) \in \mathrm{M}(\mathrm{C}^*(\mathbb{H}))$.

Theorem C.2. *Let a be the complex generator of the algebra $\mathcal{E}$. Then a is affiliated with $\mathrm{C}^*(\mathbb{H})$.*

Proof. For any $z \in \mathbb{C}$, $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$ we set

$$h_t(z, x) = \frac{x \exp tx}{4\pi \sinh tx} \exp\left(-\frac{|z|^2 x \coth tx}{4}\right) \in \mathbb{R}_+. \tag{100}$$

Let us define an element $H_t \in \mathrm{M}(\mathrm{C}^*(\mathbb{H}))$ by the formula:

$$H_t = \int_{\mathbb{C}} d^2 z\, h_t\left(z, \frac{1}{2}\lambda\right) U_{z,0}.$$

To be more precise, one can check that for any $b \in \mathrm{C}^*(\mathbb{H})$ and $f \in \mathrm{C}_c^\infty(\mathbb{C})$ the integral

$$\int d^2 z\, h_t\left(z, \frac{1}{2}\lambda\right) U_{z,0} f(\lambda) b$$

exists in the norm sense and the following inequality holds:

$$\left\| \int d^2 z\, h_t\left(z, \frac{1}{2}\lambda\right) U_{z,0} f(\lambda) b \right\| \leq \| f(\lambda) b \| .$$

Hence H_t is well defined on the elements of the form $f(\lambda)b$ and by the above inequality it can be extended to the whole $\mathrm{C}^*(\mathbb{H})$ giving a selfadjoint element of $\mathrm{M}(\mathrm{C}^*(\mathbb{H}))$. Let us list some properties of H_t.

1. The map $\mathbb{R}_+ \ni t \mapsto H_t \in \mathrm{M}(\mathrm{C}^*(\mathbb{H}))$ is a norm-continuous semigroup and $\|H_t\| \leq 1$.
2. $\lim_{t\to 0} H_t b = b$ for any $b \in \mathrm{C}^*(\mathbb{H})$.
3. For any $b \in \mathrm{D}^\infty(U)$ the map $\mathbb{R}_+ \ni t \mapsto H_t b$ is differentiable and

$$\frac{d}{dt} H_t b \Big|_{t=0} = -a^* a\, b.$$

The point 1 of this list enables us to define the element $\Xi \in \mathrm{M}(\mathrm{C}^*(\mathbb{H}))$:

$$\Xi = \int_{\mathbb{R}_+} dt\, \exp(-t) H_t.$$

Using point 2 and 3 we can check that for any $b \in \mathrm{D}^\infty(U)$ we have $\Xi b \in \mathrm{D}^\infty(U)$ and

$$(1 + a^* a)\Xi b = b.$$

This shows that $\overline{(1 + a^* a)\,\mathrm{D}^\infty(U)}^{\|\cdot\|} = \mathrm{C}^*(\mathbb{H})$, which is sufficient for a to be affiliated with $\mathrm{C}^*(\mathbb{H})$. This last statement follows from Proposition 2.2 of [12]. □

Remark C.3. Let $\mathcal{H}$ be a Hilbert space. Analyzing the above proof one can conclude that given any representation $\pi \in \mathrm{Rep}(\mathrm{C}^*(\mathbb{H}); \mathcal{H})$, a compactly supported function $f \in \mathrm{C}_0(\mathbb{C})$ and $v \in \mathcal{H}$ we have

$$\exp(-t\pi(a)^*\pi(a))\pi(f(\lambda))h = \int_{\mathbb{C}} d^2 z\, h_t\left(z, \frac{1}{2}\pi(\lambda)\right) \pi(U_{z,0} f(\lambda))v \tag{101}$$

where the integral on the right is taken in the sense of norm topology on $\mathcal{H}$. Let us also note that given any $v \in \mathcal{H}$ such that the differential

$$\frac{\partial}{\partial z}\pi(U_{z,0})v\Big|_{z=0}$$

exists, we have $v \in \mathrm{D}(\pi(a))$ and

$$\pi(a)h = 2\frac{\partial}{\partial z}\pi(U_{z,0})v\Big|_{z=0}. \tag{102}$$

Further, let B be a C^*-algebra and $\pi \in \mathrm{Mor}(\mathrm{C}^*(\mathbb{H}); B)$. For any $b \in B$, such that the differential

$$\frac{\partial}{\partial z}\pi(U_{z,0})b\Big|_{z=0}$$

exists, we have $b \in \mathrm{D}(\pi(a))$ and

$$\pi(a)b = 2\frac{\partial}{\partial z}\pi(U_{z,0})b\Big|_{z=0}. \tag{103}$$

D. Product of affiliated elements

Let A be a C*-algebra, and $T_1, T_2 \,\eta\, A$. In general, the product of T_1 and T_2 is not well defined, but it can be defined, assuming that T_1 and T_2 commute in a good sense. The construction of the product given here is a generalization of the case when $A = A_1 \otimes A_2$, $T_1 = S_1 \otimes 1$ and $T_2 = 1 \otimes S_2$ where $S_1 \,\eta\, A_1$ and $S_2 \,\eta\, A_2$, in which the product of T_1 and T_2 is the tensor product construction $S_1 \otimes S_2 \,\eta\, A_1 \otimes A_2$ which was described in [10].

Definition D.1. Let A be a C*-algebra and let T_1, T_2 be elements affiliated with A. Let $z_1, z_2 \in \mathrm{M}(A)$ be z-transforms of T_1 and T_2 respectively. We say that T_1 and T_2 strongly commute if

$$z_1 z_2 = z_2 z_1, \tag{104}$$

$$z_1^* z_2 = z_2 z_1^*. \tag{105}$$

Let T_1 and T_2 be a pair of closed operators acting on a Hilbert space $\mathcal{H}$. We say that T_1 and T_2 strongly commute if they strongly commute as elements affiliated with the algebra of compact operators $\mathcal{K}(\mathcal{H})$.

Theorem D.2. *Let A be a* C**-algebra and let $T_1, T_2 \,\eta\, A$ be a strongly commuting pair of affiliated elements. Let us consider the set $\mathrm{D}(T_0) = \{a \in \mathrm{D}(T_2) : T_2 a \in \mathrm{D}(T_1)\}$ and define the operator $T_0 : \mathrm{D}(T_0) \to A$ by the formula $T_0 a = T_1(T_2 a)$. Then T_0 a is closable operator acting on the Banach space A and its closure T_0^{cl} is affiliated with A. This closure will be denoted by $T_1 T_2$. We also have $T_1 T_2 = T_2 T_1$.*

Proof. We define $T_1 T_2$ using the method described in Theorem 2.3 of [12]. The related matrix $Q \in \mathrm{M}(A) \otimes \mathrm{M}(\mathbb{C}^2)$ is of the form:

$$Q = \begin{pmatrix} (1 - z_1^* z_1)^{\frac12} (1 - z_2^* z_2)^{\frac12} & -z_1^* z_2^* \\ z_1 z_2 & (1 - z_1 z_1^*)^{\frac12} (1 - z_2 z_2^*)^{\frac12} \end{pmatrix}.$$

(compare with the matrix Q from the proof of Theorem 6.1 of [10]). Q satisfies all the assumptions of Theorem 2.3, hence it gives rise to an affiliated element. We leave it to the reader to check that this affiliated element is $T \in A^\eta$ of our theorem. ☐

For the needs of this paper we shall prove the following lemmas.

Lemma D.3. *Let A be a* C**-algebra, T an element affiliated with A and X a dense subspace of $\mathrm{D}(T)$. Then :*

1. *if $(1 + T^*T)^{\frac12} X$ is dense in A, then X is a core of T;*
2. *if $X \subset \mathrm{D}(T^*T)$ and $(1 + T^*T)X$ is dense in A, then X is a core of T.*

Proof. It is easy to see that for any dense subspace $X' \subset A$ the set $(1 + T^*T)^{-\frac12} X'$ is a core of T. Taking $X' = (1 + T^*T)^{\frac12} X$ we get the proof of point (1) of our lemma. To prove point (2) note that $(1+T^*T)^{-\frac12} X'$ is dense in A whenever X' is dense in A. Applying this to the set $(1+T^*T)X$ of point (2) we see that $(1+T^*T)^{\frac12} X$ is dense in A. Using point (1) we conclude that X is a core of T. ☐

Lemma D.4. *Let $T_1, T_2 \in A^\eta$ strongly commute and let $X \subset A$ be a dense subspace. Then the set*

$$(1 + (T_1 T_2)^* (T_1 T_2))(1 + T_1^* T_1)^{-1} (1 + T_2^* T_2)^{-1} X$$

is dense in A. In particular $(1 + T_1^* T_1)^{-1} (1 + T_2^* T_2)^{-1} X$ is a core of $T_1 T_2$.

Proof. Note that

$$(1 + (T_1 T_2)^* (T_1 T_2))(1 + T_1^* T_1)^{-1} (1 + T_2^* T_2)^{-1} = (1 + (T_1^* T_1)(T_2^* T_2))(1 + T_1^* T_1)^{-1} (1 + T_2^* T_2)^{-1}.$$

We express the right hand side of the above equation using z-transforms of T_1 and T_2:

$$(1 - z_{|T_1|}^2)(1 - z_{|T_2|}^2) + z_{|T_1|}^2 z_{|T_2|}^2.$$

Let $f : [0,1] \times [0,1] \to \mathbb{R}_+$ be the function defined by

$$f(x_1, x_2) = (1 - x_1^2)(1 - x_2^2) + x_1^2 x_2^2.$$

Note that $f(x_1, x_2) = 0$ if and only if $x_1 = 1$ and $x_2 = 0$ or $x_1 = 0$ and $x_2 = 1$. Let us also define the function $g : [0,1] \times [0,1] \to \mathbb{R}$ by the formula:

$$g(x_1, x_2) = (1 - x_1^2)^{\frac{1}{2}}(1 - x_2^2)^{\frac{1}{2}}.$$

We have the following implication:

$$(f(x_1, x_2) = 0) \Rightarrow (g(x_1, x_2) = 0).$$

Using Proposition 6.2 of [10] we get the following inclusion:

$$\overline{(1 + T_1^*T_1)^{-\frac{1}{2}}(1 + T_2^*T_2)^{-\frac{1}{2}}X}^{\|\cdot\|} \subset \overline{(1 + (T_1^*T_1)(T_2^*T_2))(1 + T_1^*T_1)^{-1}(1 + T_2^*T_2)^{-1}X}^{\|\cdot\|}.$$

We end the proof by noting that $A = \overline{(1 + T_1^*T_1)^{-\frac{1}{2}}(1 + T_2^*T_2)^{-\frac{1}{2}}X}^{\|\cdot\|}$. $\square$

Lemma D.5. *Let $T_1, T_2 \in A^\eta$ be a strongly commuting pair of operators andlet $Y \subset \mathrm{D}(T_2^*T_2)$ be such that $(1 + T_2^*T_2)Y$ is dense in A. Then the set*

$$(1 + (T_1T_2)^*(T_1T_2))(1 + T_1^*T_1)^{-1}Y$$

is dense in A.

Proof. The proof of this lemma follows from the previous one with $X = (1 + T_2^*T_2)Y$. $\square$

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